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PLASTIC BEHAVIOR OF ENGINEERING MATERIALS

Part 2. Partially Plastic Thick-Walled Cylinders

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Wright Air Development Center
Air Research and Development Command
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Wright-Patterson Air Force Base, Ohio

FOREWORD

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ABSTRACT

This report presents experimental and theoretical work on the overstraining of thick-walled cylinders. Four mild steel cylinders (2:1 wall ratio) were subjected to internal fluid pressure and strains at the bore and the outside surfaces were measured. In addition, the mechanism of flow was studied by polishing the end and outside surfaces for the observation of Lueders lines. A theoretical analysis is given which is based on results from a quantitative comparison of certain previous theories and available experimental data. The solution is in closed form and is applicable to strain-hardening materials.

Observations disagree with theoretical assumptions concerning the progression of yielding; wedge regions of overstrained material, occupying a small fraction of the total volume, characterize the yielding process. Discrepancies with theory are observed in the measured strains; fully plastic load-carrying capacities predicted from theory are higher than those observed in the experiments. Instability of deformation (creep) under maintained constant load is discussed.

It is concluded that theoretical analyses, in their present form, do not cope adequately with the inelastic problem concerning the wedge type of yielding in two and three dimensional, non uniform stress fields. Suggestions are given for further research.

PUBLICATION REVIEW

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$d\sigma$; $d\varepsilon$	principal stress increment; principal strain increment.
σ; ε	principal stress; principal strain.
σ_θ, σ_r, σ_z; ε_θ, ε_r, ε_z	principal stresses and strains in the hoop, radial and axial directions, respectively.
ε_e; ε_p	elastic component of strain; plastic component of strain.
$\bar{\sigma}$; $\bar{\varepsilon}$	effective stress; effective strain.
σ_e	tensile yield stress
s $(=\dfrac{\sigma_e}{2})$	yield shear stress in tension.
τ; γ	maximum shear stress; maximum shear strain.
γ_e	maximum shear strain at yield.
r_o	internal radius.
$r = y\, r_o$	variable radius.
$r_n = n\, r_o$	plastic-elastic boundary radius.
$r_1 = k\, r_o$	outside radius.
λ, $d\lambda$, β	plastic flow functions.
P	internal pressure.
G; E	modulus of rigidity; elasticity.
m	strain hardening parameter.
ν	Poisson's ratio.
$\varepsilon_\theta)_{o/d}$, $\varepsilon_\theta)_{bore}$	Circumferential strains at outside and bore surfaces respectively.

1. Introduction.

The partial overstraining of thick-walled cylinders by internal uniform pressure has claimed the attention of several prominent investigators. Table I lists some of the theoretical contributions and illustrates differences in basic assumptions. The rigorous mathematical solutions require numerical procedures for their completion and, in general, the solutions are so complex that little practical use may be made of them by design engineers.

A solution to the problem must satisfy certain conditions, some of which are not unique but possess justifiable alternatives. Different combinations of these conditions account for the various theories in Table I. An outline of the requirements for a solution will help to clarify the issues and also provide a basis for understanding the need for the investigations reported here.

(a) Radial Equilibrium

$$\frac{\partial(y\sigma_r)}{\partial y} = \sigma_\theta \tag{1}$$

(b) Strain Compatability

$$\frac{\partial(y\epsilon_\theta)}{\partial y} = \epsilon_r \tag{2}$$

(c) Boundary Conditions

$$\sigma_r = -P \ , \ y = 1$$

$$\sigma_r = 0 \ , \ y = k$$

Either, $\epsilon_z = 0$

TABLE I. - PARTIALLY-PLASTIC THICK-WALLED CYLINDER THEORIES

Author	Plastic Stress-Strain Law	Compressibility		Flow Condition	Axial Boundary	Type of Solution
		Plastic Domain	Elastic Domain			
Hill, Lee, and Tupper [1]	Prandtl-Reuss	Comp.	Comp.	Tresca	$\epsilon_z = 0$	Numerical
Hill, Lee, and Tupper [2]	Prandtl-Reuss	Comp.	Comp.	Tresca	Closed Ends	Numerical
MacGregor, Coffin, and Fisher [3]	Hencky	Comp.	Comp.	Von Mises	Open Ends	Numerical
Sokolovsky [4]	Hencky	Comp.	Comp.	Von Mises	$\epsilon_z = 0$	Numerical
Sokolovsky [4]	Hencky	Comp.	Comp.	Von Mises[a]	$\epsilon_z = 0$	Numerical
Sokolovsky [4]	Hencky	Incomp.	Comp.	Von Mises[a]	$\epsilon_z = 0$	Closed Form
Hodge, and White [5]	Prandtl-Reuss	Comp.	Comp.	Von Mises	$\epsilon_z = 0$	Numerical
Hodge, and White [5]	Hencky	Comp.	Comp.	Von Mises	$\epsilon_z = 0$	Numerical
Allen and Sopwith [6]	Hencky	Comp.	Comp.	Tresca	Either $\epsilon_z = 0$ or, Closed Ends or, Open Ends	Closed Form
Cook [7]	Hencky	Incomp.	Comp.	Tresca with Upper and Lower Yield	Closed Ends	Closed Form[b]
Nadai [8]	Hencky	Incomp.	Comp.	Von Mises	$\epsilon_z = 0$	Closed Form
Steele [9]	Hencky	Incomp.	Comp.	Tresca	Either $\epsilon_z = 0$ or, Closed Ends or, Open Ends	Closed Form
Steele [10] (Theory of this report)	Hencky	Incomp.	Comp. or Incomp.	Tresca[a]	Either $\epsilon_z = 0$ or, Closed Ends or, Open Ends	Closed Form

[a] Inclusion of Strain Hardening; The Non-asterisked Solutions Assume the Material to Flow According to the Function Defining the Theory of Failure.

[b] Solution Dependent on Experimental Measurements which Cook Made on Closed-Ended Cylinders.

$$\text{or,} \int_{1}^{k} \sigma_z y \, dy = 0 \quad \text{(open ends)} \qquad (3)$$

$$\text{or,} \int_{1}^{k} \sigma_z y \, dy = \frac{P}{2} \quad \text{(closed ends)}$$

Choosing the axial strain as zero is convenient mathematically and is popular for the more rigorous solutions. It is a condition, however, not found in practice where cylinders have either open or closed ends. Solutions for $\epsilon_z = 0$ and closed ends do not differ greatly, but the difference is larger for open-ended cylinders.

    (d) <u>Plane Sections to remain Plane</u>

        i.e. ϵ_z is independent of r.

    (e) <u>Continuity across Elastic-Plastic Interface</u>

In all theoretical solutions the elastic-plastic boundary is taken to be circular and the cylinder to be comprised of two regions, an inner plastic and an outer elastic one, both of homogeneous material. Continuous displacement across the elastic-plastic interface implies continuous ϵ_θ and ϵ_z. Equilibrium demands continuous σ_r. Conditions at the elastic-plastic boundary may be deduced from these. For example, if incompressibility of the plastic material only is assumed then ϵ_r cannot be continuous (see eqs. 7 and 8). Also, if the tensile stress-strain relation is discontinuous (e.g., upper and lower yield points) then σ_θ must be discontinuous if the maximum shear stress yield condition is used.

The yielding of mild steel is characterized by the formation of wedge-shaped veins of overstrained materials,[11] and it is doubtful whether theories, assuming circular elastic-plastic boundaries, can describe adequately the stresses and deformations in a mild steel cylinder. However, the alternative problem of analyzing the effects of such wedge regions of

plastic material embedded in an elastic medium has so far proved exceedingly difficult.

(f) Plastic Stress-Strain Relations

One of two plastic stress-strain relations replaces the well-known Hooke's law for elastic bodies. The Prandtl-Reuss[12] relations provide the basis for the "incremental strain" or "flow" type of theory. They take the form

$$E\,d\varepsilon_\theta = d\sigma_\theta - \nu d\,(\sigma_r + \sigma_z) + d\lambda\left[\sigma_\theta - \frac{1}{2}\,(\sigma_r + \sigma_z)\right] \qquad (4)$$

(plus two similar equations for the principal strain increments in the radial and axial directions).

The Hencky[13] relations describe the "total strain" or "deformation" theory. They may be written

$$E\,\varepsilon_\theta = \sigma_\theta - \nu\,(\sigma_r + \sigma_z) + \lambda\left[\sigma_\theta - \frac{1}{2}\,(\sigma_r + \sigma_z)\right] \qquad (5)$$

(plus two similar equations for the principal strains in the radial and hoop directions).

The basic differences in eqs. (4) and (5) have been adequately described elsewhere[13]. It is well established that the incremental theory is the more correct, and that, if the principal axes of stress remain fixed in direction, and the principal stress ratios are constant, the Prandtl-Reuss and Hencky relations yield the same result. Frequently, even though the conditions on stress axes and ratios are not strictly maintained, solutions using eq. (5) vary only a small amount from those solved by the Prandtl-Reuss relations. This is the case for partially plastic thick-walled cylinders subjected to internal uniform pressure only, and the small inaccuracy in using the Hencky relations is compensated for by certain advantages of mathematical convenience.

(g) Compressibility of Plastic Material

The total strain in an overstrained material is composed of an elastic and a plastic component; thus

$$\varepsilon = \varepsilon_e + \varepsilon_p \tag{6}$$

It is observed from experiments on metals that there is no change in volume due to the plastic component of strain. Thus volume changes during plastic deformation are elastic; hence

$$\varepsilon_\theta + \varepsilon_r + \varepsilon_z = \frac{1 - 2\nu}{E} (\sigma_\theta + \sigma_r + \sigma_z) \tag{7}$$

If the stresses and strains obey eq. (7), the material is said to be <u>compressible</u>. If the plastic component of strain is predominant

$$\varepsilon_\theta + \varepsilon_r + \varepsilon_z \approx 0 \tag{8}$$

and the material is said to be <u>incompressible</u>. In the cylinder problem under consideration the plastic material is restrained by an outer elastic hoop. The plastic strains are thus comparable in magnitude to the elastic ones and errors are introduced if eq. (8) is used. The inclusion of compressibility, however, considerably complicates a solution. It is seen from Table I that there is only one solution in closed form observing compressibility of the plastic material.

(h) Flow condition

The theories of failure due to Tresca (maximum shear stress) and Von Mises (maximum energy of distortion) are well known. They are usually stated as

$$\sigma_\theta - \sigma_r = \sigma_e = 2s \tag{9}$$

where,
$$\sigma_\theta \geqslant \sigma_z \geqslant \sigma_r$$

and,
$$(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2 = 2\sigma_e^2 \tag{10}$$

Most cylinder theories incorporating strain-hardening make use of a generalization of eq. (10). The "effective stress" and "effective strain" are defined as

$$\bar{\sigma} = \frac{\sqrt{2}}{2} \sqrt{(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2}$$

$$\text{and,} \quad \bar{\epsilon} = \frac{\sqrt{2}}{3} \sqrt{(\epsilon_\theta - \epsilon_r)^2 + (\epsilon_r - \epsilon_z)^2 + (\epsilon_z - \epsilon_\theta)^2}$$

(11)

and it is assumed that $\bar{\sigma} = f(\bar{\epsilon})$ for all combinations of stress. Hill[14] has pointed out the limitations of this assumption. As for the case of the Hencky stress-strain relations, however, this strain-hardening function has approximate application in many instances and in particular to the problem under consideration. It is not common to make use of an extension of eq. (9) for strain-hardening material. Ludwik[15] in 1909, stated that the maximum shearing stress as a function of the maximum shearing strain described the subsequent flow of a ductile material in the strain-hardening range. Little use has been made of this assumption.

Clearly, the decision as to which failure condition and subsequent flow law to use, is dependent on the experimental behavior of the cylinder material. Tests by Davis[16] on low-carbon steel thin-walled cylinders under combined stress have shown that Von Mises' theory closely describes failure and subsequent flow condition. In steel with a drop in stress at yield point, Morrison[17] states that initiation of yield is due to a critical maximum shear stress but that plastic flow continues in accordance with Von Mises' criterion. The latter tests have all been made on specimens in which there exist uniform stress fields, (uniformly distributed stress) and the application of the results, in some instances, to non-uniform stress fields, requires considerable caution. For example, the experimental

work of this report (also ref. (18) on low-carbon steel and of Macrae[19] on nickel-gun-steel thick-walled cylinders, contradicts the findings of the previous results from thin-walled cylinder tests. This is attributed to the mechanism of flow in steels containing a flat-topped portion at yield in a tension test.

Cook[7] subjected mild-steel, thick-walled cylinders of various wall ratios to internal fluid pressure and observed the changes in outside diameter, circumferentially and axially. Tensile tests of the material were made on carefully prepared specimens and tested in a special machine which recorded upper and lower yield points. His cylinders were also prepared carefully and the results showed the counterpart of the upper and lower yield stress in a tension test. Discrepancies from theory were noted, and their cause was attributed to the existence of this latter phenomenon. The existence of an upper and lower yield point for mild steel material is well established, but it is considered unreasonable to expect any close correlation to occur between a tension test and a thick-walled cylinder. For the magnitude of the upper yield point in a tension test is dependent upon many factors (e.g., testing machine, surface finish, specimen shape), and such conditions are difficult to reproduce in the testing of a cylinder. In addition, the stress fields are different, and these must have a bearing on the stability of the sudden deformation. It is felt, that the careful preparation of Cook's specimens, masked the true nature and effects of the mechanism of yield. Muir[20] has deduced, and the authors will show this to be correct, that an overstrained cylinder of mild steel is composed mostly of elastic material with only a small volume of plastic material in the form of wedges. His argument was qualitative at the time, as the extent and number of wedges were not known. Accepting the above concept of yielding, it would be expected that the records of changes at the outside diameter, a position remote from the

yielding material, would disagree with a theory based on homogeneous yielding. The measurement of changes at the bore surface, where yielding initiates, would help to clarify the situation.

The experimental part of this report is thus twofold: (i) To investigate the mechanism of yielding in mild steel cylinders and to compare it with the conventional assumptions of the theories of plasticity; it should be emphasized that the results obtained need not be restricted to thick-walled cylinders only, but should provide a description of flow, characteristic of all members in which the stresses are non-uniformly distributed.

(ii) To measure strain at the bore and outside surfaces and estimate the nature and extent of deviations from theoretical values. In addition, the highly important problem to the engineer, of instability of deformations (creep) under maintained loads, is brought to the fore.

The theoretical part of the report is aimed at providing a reasonably uncomplicated solution, based on sound simplifying assumptions and available experimental data, for cylinders made of material which exhibit strain-hardening. The theory can also be used for materials with no strain-hardening in the range of strains encountered in a partially plastic cylinder.

2. Experimental Work

(a) Experimental Equipment.

(i) Pressure Apparatus

The apparatus is designed to apply internal fluid pressures to open-ended thick-walled cylinders. In principle, a fluid is confined inside the test cylinder and compressed by means of a plunger and loading machine.

Fig. 1 shows the assembled arrangement in section. It consists of a central core A which fits inside the test cylinder. The core is built in two parts which enables it to be fitted without interference to electrical strain gauges on the bore surface of the cylinder. Synthetic rubber O-ring seals B prevent leakage at the ends. A plunger C operates in a fitted hole and connecting holes D allow the system to be filled with fluid. A load on the plunger compresses the fluid which transmits a pressure to the inside of the test cylinder. An O-ring seal on the plunger and a neoprene washer E at the screw thread ensure a leakage free system. F is a pressure gauge pipe connection of the Bridgman "unsupported area" type. At G two of eight units are shown (two per flat). They are set in the core to convey electrical leads out of the pressure area, and are shown in an adjoining detail. The load is transmitted from a compression machine to the plunger via a spherical head H.

The test cylinder moves axially against a small amount of friction at the O-rings, nevertheless, the open-ended condition of testing is closely assimilated. The ratio of cylinder length under pressure to total cylinder length is 0.9375. The system is leakage free and sensitive control of the application of pressure is a feature of the apraratus.

(ii) Measuring Devices.

A 25,000 lbs. per in.^{2} Bourdon Gauge measured pressure. Its smallest division reads 100 lbs. per in.^{2} and an estimation correct to 20 lbs. per in.^{2} is obtainable. The gauge was calibrated against a dead weight

calibrator before and after each test.

Electrical resistance gauges were used in the measurement of strain. Records were made on a standard Young bridge unit. Paper backed gauges of one-inch gauge length were used on the outside diameter of the test cylinder. Preliminary work on paper gauges under fluid pressure gave unreliable results, hence bakelite gauges of one-quarter inch-gauge length were used on the bore surface. Standard procedures were adopted for the mounting of all gauges.

It was found that no protection was necessary for the internal gauges from the fluid pressure. The effect of pressure on them was studied in preliminary work and is given in Appendix I. This work consisted of mounting gauges on a rectangular strip and immersing it in the fluid under pressure. The effect of pressure was found to be small and may be allowed for by assuming Lame's theory for the elastic strains to be correct.

Mounting of bakelite gauges requires a curing sequence consisting of a temperature treatment under a normal pressure of from 100 to 200 lbs. per in.^{2}. This pressure was effected by the spring loaded device shown in Fig. 2. Neoprene pads fastened to metal shoes containing a spring ensure the necessary uniform normal pressure. This device assisted also in the accurate positioning of the gauges in the cylinder.

In three of the cylinders the end faces were polished for the observation of Lueder's lines. One of the cylinders was polished on the outside surface also.

(iii) Specimens.

Tests were performed on four cylinders (Fig. 3). The material was mild steel and specimens were rough machined from a hot-rolled billet, annealed at 1630° F. and cooled in the furance. They were finish machined to size by a sharp tool and adequate lubricant.

Fig. 4a shows the position of cylinders taken from the billet and the material for tension tests. Tensile specimens were taken from positions (Fig. 4b) corresponding to the bore layers in the test cylinders. They were given the same annealing treatment. Average material characteristics are shown in Table (2) for four positions in the billet.

Billet position numbers from the left. (See Fig.4(a))	YOUNG'S MODULUS (10^6 lbs. per in.^{2})	YIELD STRESS (= 2 · YIELD SHEAR STRESS) (lbs. per in.^{2})
1	28.97	25,000
2	29.6	32,500
3	28.9	32,000
4	30	32,500

Table 2 - AVERAGE MATERIAL PROPERTIES FOR

4 POSITIONS IN THE BILLET.

The stress-strain diagrams in tension have a sharply defined yield point as shown in Fig. 4c.

Polishing of the cylinders for the observation of Lueder's lines consisted of grinding followed by finishing with 00 emery cloth.

(b) Test Procedures

Readings of strain were recorded every 500 lbs. per in.^{2} in the early part of elastic straining and 200 lbs. per in.^{2} close to the initiation of yield and in the subsequent progression of yield wedges to the outside surface. Each pressure was held for at least 15 minutes and in some cases longer when creep was found to be operative. For example in cylinder No. 4, the maximum pressure was held for 100 minutes to study more fully the effect of creep.

Cylinder No. 1.

Four circumferential gauges were placed symmetrically round the central surface of the bore. Four pairs of circumferential and axial gauges were placed in corresponding positions on the outside. This cylinder was the first to be tested and the end faces were not polished for the observation of Lueder's lines. The fully plastic value of pressure was reached in one loading cycle.

Cylinders No. 2, 3, and 4.

Gauges were affixed at six symmetrical positions as shown in Fig. 3. The end faces were polished and in the case of cylinder No. 4, the outside surface also. Cylinder No. 2 was loaded, unloaded, and then reloaded three times, each loading reaching a higher pressure and causing more yielding before the wedges reached the outside. No. 3 was taken to the fully plastic value in one cycle of loading. No. 4 cylinder was loaded to give a theoretical depth of yield corresponding to slightly more than one-half the wall thickness. The apparatus was then disassembled and the Lueder's lines observed. Subsequently, the cylinder was reloaded to cause the yield wedges to penetrate to the outside and the Lueder's lines again observed.

Cylinder No. 3 had additional gauges placed at the quarter points, (see Fig. 3). This was to decide if the cylinder length to diameter ratio was large enough to eliminate end effects at the measuring sections.

(c) Results

The results obtained from the tests fall into two categories - quantitative recordings of strain against pressure, and quasi-quantitative observations of the mechanism of yielding by means of Lueder's lines. The original strain records are presented as graphs of pressure against strain in Figs. 5(a to d). Average values of bore, outside circumferential, and

axial strains are plotted non-dimensionally in Figs. 8, 9, and 10, respectively. It will be noted that the bore gauges for cylinders No. 3 and 4 failed at comparatively low pressures. This is attributed to a loosening of the bond due to using a faulty cement. The bakelite cement deteriorates with age. One or two of the outside surface gauges were also found to be faulty.

Lueder's lines were noted by visual examination under oblique lighting, and carefully traced by means of a black crayon. The lines could then be photographed or copied on to tracing paper (Fig. 6). An indication of the manner in which the cylinder yields is given in Fig. 7, where the variation of circumferential strains is shown corresponding to wedge regions for two of the cylinders.

Recordings of increase of strain at constant pressure (creep) were taken as additional to the immediate purpose of the work, and some of these records are shown in Fig. 11.

(d) Discussion of Results

    (i) Measurement techniques.

A technique was developed for the measurement of bore strains in thick-walled cylinders. The effect of pressure on the gauges was deduced by assuming Lame's theory to be correct. This effect is found to be so small that it may be ignored. Consequently, all the graphs are plotted without any correction for pressure. The technique is further verified by the consistent behavior of gauges on subsequent re-loadings. In general the gauges fail at large strains at the bore surface in a decisive manner at a value of pressure causing approximately the full yielding of the cylinder. As is pointed out in a previous section, however, this failing was premature for cylinders Nos. 3 and 4 due to the use of a deteriorated cement.

The non-dimensional theoretical plot for the theories assumes the

cylinder boundaries to remain circular and consequently provide a pressure versus change in length relationship. The question arises, therefore, whether the average of the several strain recordings adequately describes the average change in a length dimension. This is only pertinent for a material such as mild steel where adjacent regions carry widely different strains due to the occurrence of Lueder's bands. Load-deformation tests have been performed in the Department of Theoretical and Applied Mechanics, University of Illinois, on mild steel beams[21] with gauges of different gauge lengths yet covering the same overall dimension. It has been shown that the average of several gauges of small length agree with observations made over the overall dimension, even though wide variations exist among the smaller gauges. It is desirable, of course, to obtain a true statistical mean, to place as many gauges as possible round the circumference of the test cylinder. Space limitations allow six (1/4 inch gauge length) at the most for the size of cylinders used. It is considered that four gauges are on the lower limit for obtaining a true statistical mean but six are adequate. Cook[7] measured outside diameter changes in two positions at right angles and considered this to be satisfactory. One of the authors in some previous work[9] compared the average results from four electrical gauges on the outside with mechanical measurements made at two positions at right angles. Good agreement was found and hence the measurement of average strain by four electrical gauges placed equidistantly round the circumference was assumed to be sound.

In cylinder No. 3 gauges were located at the quarter sections along the length (Fig. 3). The results for these gauges are not presented but they showed exactly the same strain readings (within the limits of experimental error) as the corresponding ones at the central section. It is deduced therefore that the cylinder length to diameter ratios are large enough to eliminate end effects at the measured sections.

(ii) The Mechanism of Yielding.

It is seen from Figs. 5 (a t. d) that large variations in strains occur circumferentially. This is due undoubtedly to the existence of an asymmetrical distribution of wedge regions of overstrain. They may be studied from Fig. 6 and several important features are recognizable.

It is of practical and theoretical interest that a "fully plastic"[*] condition for the cylinders is obtained with only a small volume of the material actually in the overstrained condition. This is considered important because it would be unreasonable to suggest that conventional plastic theories, based on isotropic material properties, can adequately describe the stresses and strains existing in the cylinders. It is customary in theoretical treatments to assume a Prandtl-Reuss material (flat-topped tensile stress strain curve) for mild steel, at least for small strains. The very existence, however, of the flat-topped curve leads to the formation of wedge regions of overstrain (Lueder's bands) and then the stress-strain relations based on isotropy become invalid.

The distribution of wedges is asymmetrical. Hence it may be deduced that the onset of yielding at the weakest point, will destroy the symmetry of the arrangement and effect the yielding at other positions. If symmetry of shape and material were possible then initial yield would commence at all places round the inner boundary and the wedges would take the form of a symmetrical orthogonal family of logarithmic spirals. It is seen that the wedges do form logarithmic spirals since the tangent to them at any point is at a constant angle to the radius to that point. The constant angle is approximately $45^\cap$ and the wedges obviously progress along planes of maximum shear stress.

It is noticed that the position of the wedges on the top and bottom

[*] "Fully-plastic" pressure for a mild steel cylinder is defined as that pressure causing one or two wedge regions to reach the outside surface. At this value, the pressure versus circumferential strain curves become parallel to the strain axis.

faces of the cylinder correspond. This is not too obvious from cylinder No. 2 since, as can be seen from the position from which it is taken in the billet (Fig. 4a), a variation in yield stress must be presumed along its length (Table 2). In general, however, the wedges extend along the length of the test cylinder from face to face. In cylinder No. 4, the wedges can actually be seen (Fig. 6) extending along the length on the outside surface in three places.

Fig. 7 illustrates the comparison of wedge regions with the measured strains. It is seen that the inner and outer boundaries take on a complex shape due to the asymmetrical nature of the straining. In addition an important feature is shown in cylinder No. 4. There is preference for the straining due to additional load to be taken up by the propagation of originally formed wedges rather than the formation of new ones. This would not necessarily be the case if a constant pressure is maintained for hours or days on end. It is reasoned that an increase in pressure is necessary to cause further propagation of wedges once they have stopped. The formation of new wedges, however, under constant load have been observed in mild steel beams by Sidebottom, Corten, and Clark[22] several hours after the steady load was applied. It is concluded that the important problem of stability under a maintained load requires further investigation.

(iii) Comparison with Plastic Theories.

The plastic theories shown by full lines in Figs. 8, 9, and 10 are due to MacGregor, Coffin and Fisher[3] based on maximum energy of distortion flow condition and to Allen and Sopwith[6] using maximum shear stress. Both assume rotationally symmetric elastic-plastic boundaries, Prandtl-Reuss material, and isotropy in the two regions. In plotting the curves Poisson's ratio of 0.3 was used.

Good agreement is shown for the bore circumferential strains in the elastic region and the early regions of overstrain. Cylinders No. 1 and 2

yield initially at a value of pressure computed on the basis of Tresca's theory of failure (maximum shear stress). This is not the case for cylinder No. 3. It is reasoned that the very nature of the yielding mechanism precludes the accurate determination of the pressure at which yield commences. The overall results, however, seem to favor a yield condition based on Tresca. However, if reference is made to the outside circumferential observations (Fig. 9) one is misled by an apparent linearity up to pressures even exceeding the yield condition proposed by Von Mises (maximum energy of distortion). This is especially true of cylinder No. 1 where the strains continue to lie on the curve of MacGregor, Coffin, and Fisher until the wedge regions predominate and finally on reaching the outside of the cylinder conform more to the theoretical stiffness given by Allen and Sopwith.

The maximum values of pressure fall short of those predicted in Allen and Sopwith's theory and considerably more so when compared with the theory of MacGregor, Coffin, and Fisher. Thus, it must be concluded that invalid theoretical assumptions do not describe adequately the behavior of mild steel cylinders. It is of practical importance that deviations from theory are on the unsafe side.

It is seen from Fig. 10 that axial strains show large discrepancies with theory. This is to be expected in a cylinder comprised largely of elastic material with only a small volume plastic in the form of wedges. The elastic strains lie to the left of the Lame line and this may be attributed to a small amount of friction at the O-ring seals. The magnitude of the axial strains is small and the maximum deviation from Lame is of the order of 14 micro inches per inch.

(iv) Time Effects.

Creep of the strains under constant pressure is an important additional feature of the tests. After each pressure increment the load was maintained

constant until creep was not apparent. The time on the average was 15 minutes, but in some cases, especially near the fully plastic value, this time was longer. Once the wedges reach the outside surface of the cylinder the creep persists over comparatively large periods of time. This is exemplified in Fig. 11 where it is seen that at the largest pressure ($\frac{P}{S}$ = 1.315) in cylinder No. 4 the outside strains were still increasing after 100 minutes. Plots are also shown for a cylinder No. 1 gauge ($\frac{P}{S}$ = 1.36) indicating increases with time of 20 $^o/\cap$ and 23 $^o/o$ for bore and outside circumferential strains respectively. As far as can be gathered from this observation over a period of 30 minutes the deformations are stabilized but additional tests are an urgent necessity to estimate these effects over longer periods of time, even days or weeks. It may also be observed from the strain figures that at the lower pressures small creeps occur at the bore with little effect on the outside measurements. This would be important in an interference fit assembly in which the interference pressure is dependent on the bore deformations of the scantling.

3. Theoretical Work.

The mathematicians approach to the problem under consideration is fundamentally different from that of the engineer. The former is interested in an accurate knowledge of the stresses and strains in terms of the applied load or loads for all points in an idealized body. In general, the engineer seeks an estimation of the variation of factors with applied loads, which are only of significance in the functional design of the part. The rigorous solution due to Hill, Lee, and Tupper[1] shows that, in Nadai's[8] approximate solution, the axial stresses are in error by as much as 66 2/3 percent. This factor may or may not be of interest to the engineer. For materials exhibiting flat-topped (well defined yield point) stress-strain curves or small amounts of strain-hardening, the engineer is interested in deformations rather than stresses. In the case of the thick-walled cylinder, interest lies in deflections of bore and outside surfaces for specific internal pressures. An accurate knowledge of the stresses would be required only if rupture theories were to be considered before the deflections become the governing feature in design. The axial stress is small, in general, compared to the other two principal stresses, and a considerable percentage error in the axial stresses may be tolerated if their effect on the bore and outside deflections is small.

The necessity for accurate solutions to the thick-walled cylinder is not denied. Indeed history has shown that present day elastic mechanics of materials could not have been so soundly formulated without the solutions of the theory of elasticity on which to base it. It is believed that a similar evolution must take place in the case of plasticity if the advantages it has to offer are to be understood and used widely by industry. The thick-walled cylinder under internal pressure has been copiously treated by the methods of the mathematical theory of plasticity and the author feels particularly fortunate in having these solutions at his disposal for

an attempt at formulating the problem on a "plastic mechanics of materials" basis.

The first step in developing the theory will be to compare certain of the analyses listed in Table I in an attempt to estimate the effects on significant factors in design (i.e. bore and outside diameter deflection), of the various basic assumptions.

(a) Comparison of Previous Theories.[9]

Previous comparisons, with one exception[5], have involved several variables (e.g. axial boundary, compressibility, theory of failure, etc.) and the effects of the individual variable have been masked. It is proposed to adopt a standard set of conditions and examine the effect of compressibility and theory of failure on the significant stresses and strains. In addition the effect of making Poisson's ratio 0.5 rather than the normal value (0.3 for steel), is discussed for theories which assume incompressibility of the plastic material.

Hodge and White[5] have given a quantitative comparison of the Reuss stress-strain relations and the Hencky relations for cylinders in plane strain, and composed of compressible material obeying Von Mises flow condition. Their results show that the two theories yield almost identical values for the stresses and strains. It is realized that this cannot be strictly generalized to the cases of open and closed cylinders. No solution is known to the author which includes the incremental theory, compressibility, and Von Mises flow condition for closed or open ended cylinders. Incremental solutions using Tresca theory all yield the same values for hoop and radial stresses since the equilibrium equations may be directly integrated (for zero strain hardening); the solution for these stresses is statically determined. It is difficult to estimate the errors involved in the computed strains by using the Hencky stress-strain relations for open ended cylinders rather than the Prandtl-Reuss equations, but there seems at present no alternative to the selection of the former

relations for the basis of a practical solution.

A quantitative comparison follows to examine the effect of (i) compressibility and (ii) theory of failure. The basis of the comparison is:

(i) Hencky (Total Strain) plastic stress-strain relations.

(ii) Material exhibits idealized, flat-topped tensile stress-strain curve.

(iii) Open-ended cylinders.

(iv) Diameter ratio $(k) = 2$.

(v) Stresses plotted for cylinders yielded half-way through the wall $(n = 1.5)$.

The comparison is effected with reference to three theories, viz. those of (i) Allen and Sopwith[6], (ii) MacGregor, Coffin, and Fisher[3], and (iii) Steele[9]. Information on these theories may be obtained from Table I. It is seen that if Allen and Sopwith's and Steele's work are compared, the effect of compressibility may be estimated with all other variables standardized. Similarly if Allen and Sopwith's theory is compared with MacGregor, Coffin, and Fisher's solution, the differences in the stresses and strains for the Von Mises and Tresca flow condition are found.

Figs. 12, 13, and 14 show plots for these three theories of the principal stresses, and, strains at the outside and bore surfaces.

With regard to compressibility versus incompressible (plastic material) solutions for the stresses, it is seen that the only difference lies in the axial direction. In Steele's solution a discontinuity arises at the plastic-elastic boundary when Poisson's ratio for elastic material is taken as 0.3, since $\varepsilon_\theta + \varepsilon_r + \varepsilon_z = \dfrac{1 - 2\nu}{E} (\sigma_\theta + \sigma_r + \sigma_z)$ on the elastic side, and $\varepsilon_\theta + \varepsilon_r + \varepsilon_z = 0$ on the plastic side. This would be considered serious if an accurate knowledge of the axial stresses is required.

However, as pointed out on page 19, the deflections of outside and bore surfaces of the cylinder are of greater interest to the engineer and Figs. 13 and 14 show that the percentage differences here are small. Indeed the plots for outside circumferential strain for Allen and Sopwith's and Steele's theories cannot be differentiated on the scale used. The largest percentage difference is noticed in the axial strain which has little effect on the circumferential strain. To eliminate the physically unrealistic discontinuities in axial stress and bore circumferential strain, it has been suggested by Nadai (ref. 8, page 460) and Prager[23] that the elastic material should be considered incompressible also (i.e., $\nu = 0.5$). The stresses and strains for $\nu = 0.5$ are shown plotted in Figs. 12, 13, and 14 and it is seen that the curves show <u>closer</u> agreement with the more correct theory of Allen and Sopwith. Largest differences again occur in axial stress and strain but the outside and bore circumferential strains vary at the most by 3 percent for a specified applied load once the plastic region has extended to a depth of one-half the wall thickness or greater. Thus, it is contended that a solution based on incompressibility in both the plastic and elastic domains will be of value to the engineer since he is interested in factors which are not seriously affected by such radical assumptions. It should be emphasized that the foregoing is not a general statement applicable to all problems in plasticity, but one suitable for a <u>plastic</u> mechanics of materials solution for this specific problem.

Theories of failure are compared by two rigorous solutions (Allen and Sopwith, and MacGregor, Coffin, and Fisher). The plots of Figs. 12, 13, and 14 show large differences in the computed stresses and strains[*] (with the exception of the axial stress), and decision as to which theory

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[*] It should be noted that the Mises-Tresca comparison is based on the assumption that the two laws agree for a tension test. This follows the customary engineering treatment for theories of failure. However,

to use depends on the experimental behavior of the cylinder material.

It has been shown in the experimental part of this report (also ref. 18) and by Macrae[19] that, tests performed on thick-walled cylinders, gave results at variance with Von Mises theory and this was attributed to the mechanism of yielding. It is further reasoned that Von Mises theory may be operative within the wedge regions of overstrain but the integrated effect of such regions on observed strains at the outside and bore surfaces is more closely in agreement with Tresca's theory. Evidently some new theory must be devised to estimate the overall effects of wedge regions of over-strain in an elastic material, but so far this has proved exceedingly difficult.

Many materials (e.g. gun steels) have a short flat-topped portion to the tensile stress-strain curve before strain hardening becomes operative. It has been shown in the experimental work of this report that for mild steel, which has a long flat portion compared to the strain at yield, partially plastic cylinder outside and bore circumferential strains may be approximately described by Tresca's theory (zero strain-hardening). The questions now arise: What is the mechanism of flow for materials with various lengths of flat top portion, and how can the effect of the flat top and subsequent strain-hardening be accounted for in an analysis? The answers can only be given conclusively from experimental data which is not available as yet and what follows must be regarded as opinionative. It is suggested that the flat top portion gives rise to the wedge type of yielding in members in which a stress gradient exists. However, the number of wedges and extent of wedge penetration is dependent on the length of the

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(continued from previous page) the constants for the two flow conditions could equally well be chosen so that they agree for any other stress system, and if, for example, the values in pure shear (thin-walled tube under pure torsion) were made to agree exactly the difference between the two theories for the stresses and strains in the particular problem under consideration, would probably be small.

flat portion. In the limit when the flat-top portion is zero, (e.g. aluminum and aluminum alloys) so many wedges are formed that the mechanism of flow more closely represents the theoretical assumption of two regions, both of homogeneous material, separated by a circular elastic-plastic boundary. At this stage, the wedge effect on bore and outside circumferential strains disappears and a strain-hardening function obtained from experiments on a uniform, combined stress field could be applied. However, at any of the intermediate stages the effect of wedges must be taken into account with the added complication that the stress distribution within wedges may be such that the wedge material is in the strain-hardened condition. Clearly, considerable experimental data must be accumulated to clarify these opinions, and to relate constants from a simple laboratory test (e.g. tension test) to a corrected (for wedges) strain-hardening function.

It is suggested that an analysis should be based on Tresca's theory of failure as extended by Ludwik to describe the strains existing in a cylinder of strain hardening material. As a first approximation, which is deemed sufficient for partially plastic cylinders, linear strain-hardening should be considered. It is believed that this type of strain-hardening function is suitable for two reasons; firstly, it is easily handled in an analysis, and secondly it is amenable to experimental correction, to cover a wide range of materials, once sufficient data is obtained on the physical behavior of cylinders made of different strain-hardening materials.

(b) Author's Theory

Basic Assumptions

(i) $$\frac{\varepsilon_\theta - \varepsilon_r}{\sigma_\theta - \sigma_r} = \frac{\varepsilon_r - \varepsilon_z}{\sigma_r - \sigma_r} = \frac{\varepsilon_z - \varepsilon_\theta}{\sigma_z - \sigma_\theta} = \beta \tag{12}$$

β is a function of the radius only for any

stage in plastic flow. Eq. (12) together with the compressibility condition satisfy the Hencky relations (eq. (5)).

(ii)
$$\varepsilon_\theta + \varepsilon_r + \varepsilon_z = 0 \qquad (13)$$

(Incompressibility)

(iii)
$$\sigma_\theta - \sigma_r = 2f(\gamma) \qquad (14)$$

(Ludwik's strain-hardening function)

For linear strain hardening (Fig. 15),

$$\tau = s + mG(\gamma - \gamma_e) = s(1 - m) + mG\gamma \qquad (15)$$

When $m = 0$, $\tau = s$ as for zero strain-hardening.

For an incompressible material and ε_z = constant at any stage in plastic flow, compatability relations are satisfied by:

$$\varepsilon_r = \frac{\varepsilon_z}{2} - \frac{c}{r^2}$$

$$\varepsilon_\theta = -\frac{\varepsilon_z}{2} + \frac{c}{r^2}$$
(16)

where 'c' is a constant of integration.

Now
$$\gamma = \varepsilon_\theta - \varepsilon_r = \frac{2c}{r^2} = \frac{2c}{r_0^2 y^2}$$

At elastic-plastic boundary, $y = n$, $\gamma = \frac{s}{G}$.

$$\gamma = \frac{sn^2}{Gy^2} \qquad (17)$$

From Eqs. (15) and (17)

$$\tau = s(1 - m) + \frac{smn^2}{y^2} \qquad (18)$$

Radial equilibrium of an element gives:

$$y \frac{\partial \sigma_r}{\partial y} = \sigma_\theta - \sigma_r \tag{19}$$

In the elastic domain, Lame's solution[24] for Eq. (19) gives:

$$\sigma_r = A + \frac{B}{y^2}$$

$$\tag{20}$$

$$\sigma_\theta = A - \frac{B}{y^2}$$

where A and B are constants of integration.

In the plastic domain,

$$y \frac{\partial \sigma_r}{\partial y} = \sigma_\theta - \sigma_r = 2\tau = 2s(1-m) + \frac{2smn^2}{y^2}$$

On integration
$$\sigma_r = s(1-m) \log y^2 - \frac{smn^2}{y^2} + D$$

$$\tag{21}$$

$$\sigma_\theta = s(1-m) \log(y^2 + 2) + \frac{smn^2}{y^2} + D$$

where D is a constant of integration.

Boundary Conditions.

    1. $\sigma_r = 0$ at $y = k$

    2. σ_r is continuous at plastic-elastic boundary.

    3. σ_θ is continuous at plastic-elastic boundary.

Hence, in elastic domain $(n < y < k)$

$$\frac{\sigma_r}{s} = n^2\left(\frac{1}{k^2} - \frac{1}{y^2}\right)$$

$$\tag{22}$$

$$\frac{\sigma_\theta}{s} = n^2\left(\frac{1}{k^2} + \frac{1}{y^2}\right)$$

and, in the plastic domain, $(1 < y < n)$

$$\frac{\sigma_r}{s} = (1 - m) \log \left(\frac{y}{n}\right)^2 - \frac{(k^2 - n^2)}{k^2} - m\left(\frac{n^2}{y^2} - 1\right)$$

$$(23)$$

$$\frac{\sigma_\theta}{s} = (1 - m) \log \left(\frac{y}{n}\right)^2 + \frac{(k^2 + n^2)}{k^2} + m\left(\frac{n^2}{y^2} - 1\right)$$

at $y = 1$, $\sigma_r = -P$, hence,

$$\frac{P}{s} = (1 - m) \log n^2 + \frac{k^2 - n^2}{k^2} + m(n^2 - 1) \qquad (24)$$

The strain at the outside diameter is obtained from the elastic equations

$$E\epsilon_\theta = \sigma_\theta - \nu(\sigma_z)$$

substituting for σ_θ and σ_z,

$$\frac{E\epsilon_\theta)_{o/d}}{s} = \frac{2n^2}{k^2} (1 - \nu^2) - \nu \cdot \frac{E\epsilon_z}{s} \qquad (25)$$

Plastic Strains.

From Eq. (12) $\qquad \dfrac{\epsilon_\theta - \epsilon_r}{\sigma_\theta - \sigma_r} = \dfrac{\gamma}{2\tau} = \beta$

Substituting for γ and τ from Eqs. (17) and (18) respectively,

$$\beta = \frac{n^2}{2G(1 - m) y^2 + 2Gmn^2} \qquad (26)$$

Also

$$\epsilon_\theta - \epsilon_r = \gamma = \frac{sn^2}{Gy^2}$$

and, from Eq. (13), $\qquad \epsilon_\theta + \epsilon_r = -\epsilon_z$

Hence, $\qquad \epsilon_\theta = \dfrac{sn^2}{2Gy^2} - \dfrac{\epsilon_z}{2}$

$$(27)$$

$$\epsilon_r = -\frac{sn^2}{2Gy^2} - \frac{\epsilon_z}{2}$$

In particular the bore circumferential strain is given by,

$$\frac{E\epsilon_\theta)\ \text{bore}}{s} = (1 + \nu)\ n^2 - \frac{1}{2} \cdot \frac{E\epsilon_z}{s} \tag{28}$$

where $G = \dfrac{E}{2(1 + \nu)}$.

From Eq. (12), $\sigma_z = \dfrac{\epsilon_z}{\beta} - \dfrac{\epsilon_\theta}{\beta} + \sigma_\theta$ in the plastic domain.

Substituting for ϵ_θ, β, and σ_θ,

$$\frac{\sigma_z}{s} = \frac{3\left[(1 - m)\dfrac{y^2}{n^2} + m\right]}{2(1 + \nu)} \cdot \left(\frac{E\epsilon_z}{s}\right) + (1 - m)\ \log\ \left(\frac{y}{n}\right)^2 + \frac{n^2}{k^2} \tag{29}$$

also $\dfrac{\sigma_z}{s} = \dfrac{E\epsilon_z}{2} + \dfrac{2\nu n^2}{k^2}$ in the elastic region from the elastic equations.

The end condition gives:

$$\int_1^k \sigma_z y\ dy = 0\ \text{(open ends)}\ ;\ \frac{P}{2}\ \text{(closed ends)} \tag{30}$$

Assuming open ends,

$$\int_1^k \sigma_z\ y\ dy = s\int_1^n \left\{\frac{3\left[(1 - m)\dfrac{y^2}{n^2} + m\right]}{2(1 + \nu)} \cdot \frac{E\epsilon_z}{s} + (1-m)\ \log\ \left(\frac{y}{n}\right)^2 + \frac{n^2}{k^2}\right\} y\ dy$$

$$+ s\int_n^k \left\{\frac{E\epsilon_z}{s} + \frac{2\nu n^2}{k^2}\right\}\ y\ dy = 0$$

From which,

$$\frac{E\epsilon_z}{s} = -\frac{\left[(1-m)\ \log\ n^2 - (n^2-1)\left[1 - m - \dfrac{n^2}{k^2}\right] + \dfrac{2\nu n^2}{k^2}\ (k^2-n^2)\right]}{\left[\dfrac{3}{4} \cdot \dfrac{(1-m)(n^4-1)}{(1 + \nu)\ n^2} + k^2 - n^2 + \dfrac{3m\ (n^2 - 1)}{2(1 + \nu)}\right]} \tag{31}$$

Summarizing the formulae:

In elastic domain $(n < y < k)$.

$$\frac{\sigma_r}{s} = n^2 \left(\frac{1}{k^2} - \frac{1}{y^2}\right)$$

$$\frac{\sigma_\theta}{s} = n^2 \left(\frac{1}{k^2} - \frac{1}{y^2}\right)$$

$$\frac{\sigma_z}{s} = \frac{E\epsilon_z}{s} + \frac{2\nu n^2}{k^2}$$

$$\frac{E\epsilon_\theta)_{o/d}}{s} = \frac{2n^2}{k^2}(1-\nu^2) - \nu \cdot \frac{E\epsilon_z}{s}$$

In plastic domain, $(1 < y < n)$.

$$\frac{\sigma_r}{s} = (1 - m) \log \left(\frac{y}{n}\right)^2 - \frac{(k^2 - n^2)}{k^2} - m \left(\frac{n^2}{y^2} - 1\right)$$

$$\frac{\sigma_\theta}{s} = (1 - m) \log \left(\frac{y}{n}\right)^2 + \frac{(k^2 - n^2)}{k^2} + m \left(\frac{n^2}{y^2} - 1\right)$$

$$\frac{\sigma_z}{s} = \frac{3\left[(1-m)\frac{y^2}{n^2} + m\right]}{2(1 + \nu)} \cdot \frac{E\epsilon_z}{s} + (1-m) \log \left(\frac{y}{n}\right)^2 + \frac{n^2}{k^2}$$

$$\frac{E\epsilon_\theta)_{bore}}{s} = (1 + \nu) n^2 - \frac{1}{2} \cdot \frac{E\epsilon_z}{s}$$

where
$$\frac{E\epsilon_z}{s} = -\left[\frac{(1-m) \log n^2 - (n^2-1)\left[1 - m - \frac{n^2}{k^2}\right] + \frac{2\nu n^2}{k^2}(k^2 - n^2)}{\frac{3}{4} \cdot \frac{(1-m)(n^4-1)}{(1 + \nu) n^2} + k^2 - n^2 + \frac{3m(n^2 - 1)}{2(1 + \nu)}}\right]$$

also,
$$\frac{P}{s} = (1 - m) \log n^2 + \frac{(k^2 - n^2)}{k^2} + m(n^2 - 1).$$

Curves for $\frac{\sigma_r}{s}$, $\frac{\sigma_\theta}{s}$ and $\frac{\sigma_z}{s}$ are given in Fig. 16 for various values of the strain hardening parameter, m.

Figs. 17 and 18 show plots of $\frac{E\epsilon_\theta)_{c/d}}{s}$ and $\frac{E\epsilon_\theta)_{bore}}{s}$ respectively for various values of m and for $1 \leqslant n \leqslant k$.

4. Conclusions and Remarks.

(a) Relating to Experimental Work.

(i) Techniques have been developed in this investigation to measure strains at the bore surface of thick-walled cylinders and to observe the formation and propagation of Lueder's lines on the end faces and outside surfaces. They are applied to an experimental investigation of mild steel cylinders of 2:1 wall ratio.

(ii) The mechanism of yielding in mild steel is characterized by the formation of wedge regions of overstrain. The plastic material occupies a very small volume of the total. This conflicts with the conventional theoretical assumptions which predict regions of elastic and plastic homogeneous material connected by a rotationally symmetric boundary. As a consequence measured bore, outside circumferential and axial strains show discrepancies with theory.

(iii) Bore strain measurements predict initial yield to commence at a pressure calculated using Tresca's maximum shear stress theory of failure. This is not observed on the outside diameter circumferential measurements because of the aforementioned mechanism of yielding in mild steel.

Fully plastic (i.e. where the pressure versus strain curves become parallel to the strain axis, which corresponds to one or two wedges reaching the outside surface) values of pressure are lower than those predicted from Allen and Sopwith's theory and considerably in error when compared with that of MacGregor, Coffin, and Fisher. Thus it must be concluded that theory predicts unsafe values for the load carrying capacity of mild steel cylinders.

(iv) There are marked time effects (creep) which require further investigation. They are observed over periods up to 100 minutes but the author believes they should be investigated over much longer

periods of time.

(b) Relating to Theoretical Work

(i) Previous partially plastic thick-walled cylinder theories were reviewed and their various basic assumptions interpreted. It was reasoned that a search for assumptions leading to a simplified theory for design should be made from the point of view that engineers are interested in certain significant factors; for the most part, in deflections of outside and bore surfaces for specific applied pressures.

(ii) It was shown that, on the assumption of incompressibility of both elastic and plastic material, a solution for the significant factors in design is in close agreement with a more rigorous mathematical theory utilizing compressibility.

(iii) A discussion was presented of the possible mechanism of flow for strain-hardening materials with a short flat-topped portion in their stress-strain curve. It was concluded that a strain-hardening function originally proposed by Ludwik ($\tau = f(\gamma)$), was a sound basis on which to form a theory. However, at present these conclusions are opinionative and the theory must be closely supported with further experimental work on thick-walled cylinders (or any other member exhibiting a polyaxial, non-uniform stress field) made of materials with various strain-hardening characteristics.

(iv) A theory is presented, in closed form, which is based on Hencky's plastic stress-strain law, incompressibility of plastic and elastic material, and Ludwik's (linear) strain-hardening function. It furnishes a solution for closed-or open-ended cylinders.

(c) Suggestions for Further Research.

It is believed that the work recorded in this report throws new emphasis on the physical behavior of real materials stressed inelastically in two and three dimensions in a non-uniform stress field. It

should be emphasized that the thick-walled cylinder is used merely as a means to an end because of its comparative suitability to experimental and theoretical analysis. As is customary, however, the present research has unfolded new problems or placed well-known phenomena in new perspective and the following suggestions are believed justified for future work in the field.

(i) An analytical and experimental study of the stresses and strains set up in an overstrained body by the existence of wedge regions of overstrain embedded in a predominantly elastic material.

(ii) It is recognized that problem (i) is complex and even though an answer were obtained, it may not be amenable to engineering application. An alternative approach for a specific member, (e.g. a thick-walled cylinder) would be to relate experimental observations of significant factors in design, for the specific member, with an idealized theory built up from simple laboratory tests (e.g. a tension or torsion test). Both of these studies would require:

(iii) Experimental work on the type of member under consideration made of a wide range of real materials. Some data is already available but not sufficient to express views other than prophetic ones (e.g. the case of the thick-walled cylinder made of a strain-hardening material).

(iv) An experimental study of time effects (creep) for materials exhibiting the wedge type of flow. This factor is one for which engineers must have an answer before they will place any faith in the use of materials stressed above the elastic range.

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APPENDIX I

FLUID PRESSURE EFFECT ON BAKELITE GAUGES

Preliminary tests were performed to estimate the effect of fluid pressure on the bakelite gauges used to measure bore strains. They consisted of cementing three gauges onto a rectangular strip and subjecting the whole to pressures up to 25,000 lbs. per in.^{2} (maximum cylinder pressure used was 23,000 lbs. per in.^{2} approximately).

Results are presented in Figs. 19 and 20. Fig. 19 shows comparative results from two test runs for the three gauges. Fig. 20 shows results from several test runs for one gauge in order to investigate the effect of repeating the pressure cycle. Four factors emerge:-

(i) There is a small pressure effect causing a compressive strain from 2 to 5 micro inches per inch per 1000 lbs. per in.^{2}.

(ii) Small variations in pressure effect exist among gauges. These differences and also the total pressure effects are small compared to the circumferential bore strains in a cylinder of 2:1 wall ratio.

(iii) In all cases the gauges show linear response over most of the pressure range. This factor enables the pressure effect to be deduced accurately in the elastic range of the straining of a cylinder and by extrapolation for pressures causing overstraining.

(iv) There is a marked consistency for any one gauge which is subjected several times to the pressure cycle. No permanent set or creep was noticed and it may be concluded that a gauge behaves in a stable and consistent fashion under the fluid pressure.
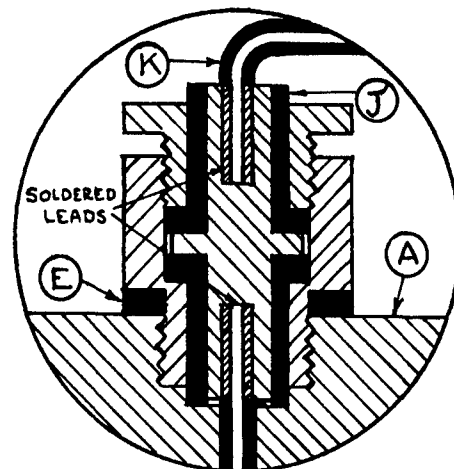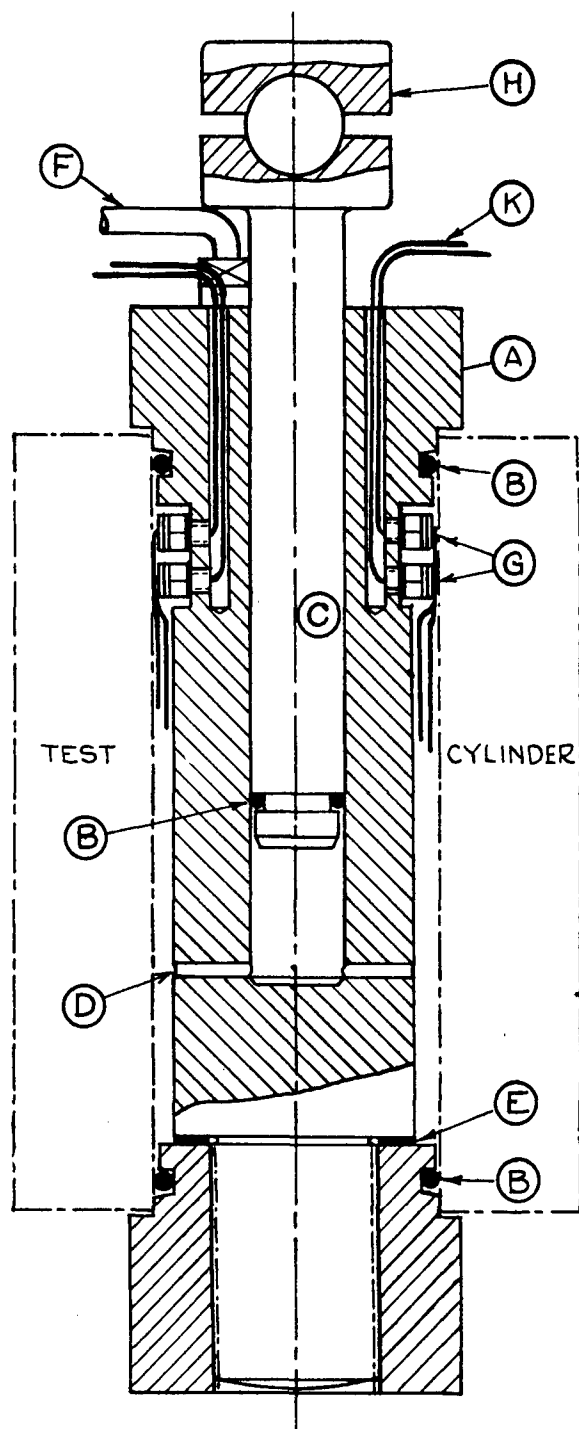
DETAIL OF ELECTRICAL
LEAD JUNCTION UNIT

(A) — CENTRAL CORE
(B) — O-RING SEAL
(C) — PLUNGER
(D) — CONNECTING HOLE
(E) — NEOPRENE WASHER
(F) — PRESSURE GAUGE CONNECTION
(G) — JUNCTION UNIT (SEE DETAIL)
(H) — SPHERICAL HEAD
(J) — EBONITE SLEEVE
(K) — INSULATED LEAD

FIG. I   SECTIONAL ARRANGEMENT OF PRESSURE APPARATUS
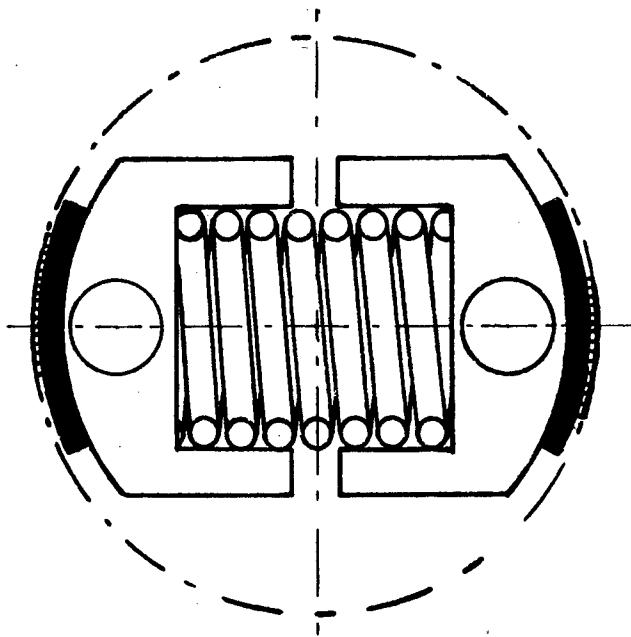
FIG. 2   SYMMETRICAL SECTION (PERPENDICULAR TO CYLINDER AXIS) OF SPRING LOADED DEVICE FOR THE MOUNTING AND CURING OF BAKELITE GAUGES
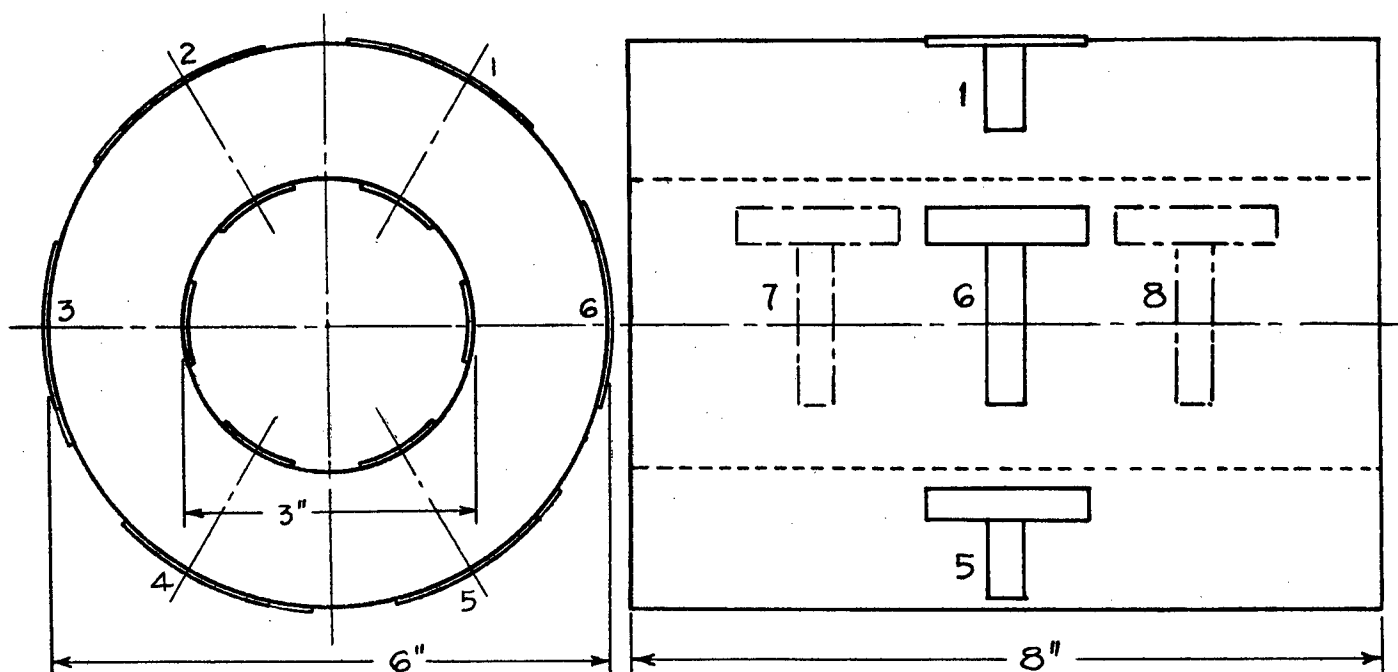
FIG. 3   TEST CYLINDER SHOWING GAUGE LOCATIONS

CYL. NO. 1   2   3   4

$6\frac{1}{2}$" D.   6" D.

$8\frac{1}{2}$"   8" $\frac{3}{4}$"

TENSILE COUPONS
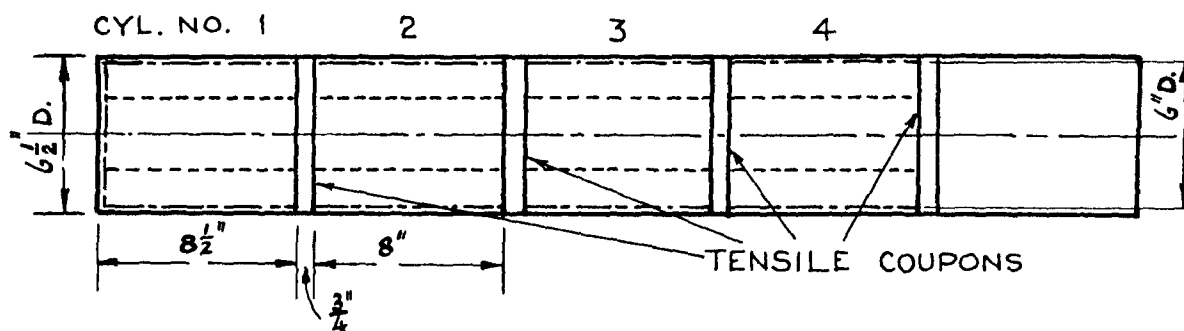
(a) POSITION OF TENSILE SPECIMENS RELATIVE TO LENGTH OF BILLET

2" $\frac{1}{2}$" D. $\frac{3}{4}$" D.
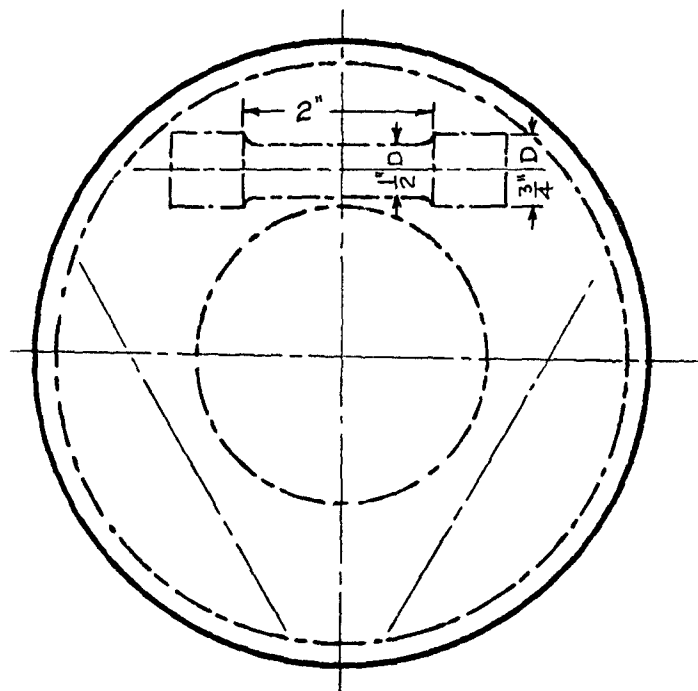
(b) POSITION OF TENSILE SPECIMENS RELATIVE TO CROSS SECTION
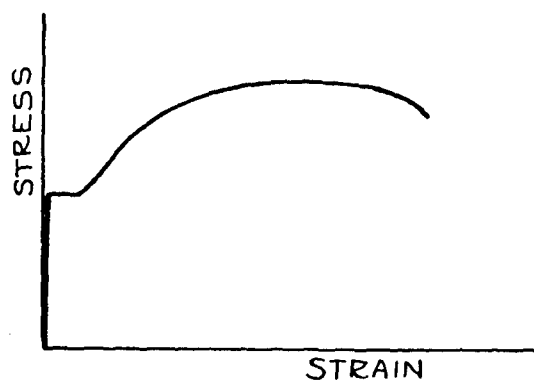
STRESS

STRAIN

(c) TYPICAL TENSILE STRESS STRAIN DIAGRAM FOR MATERIAL

FIG. 4   MATERIAL TEST DETAILS
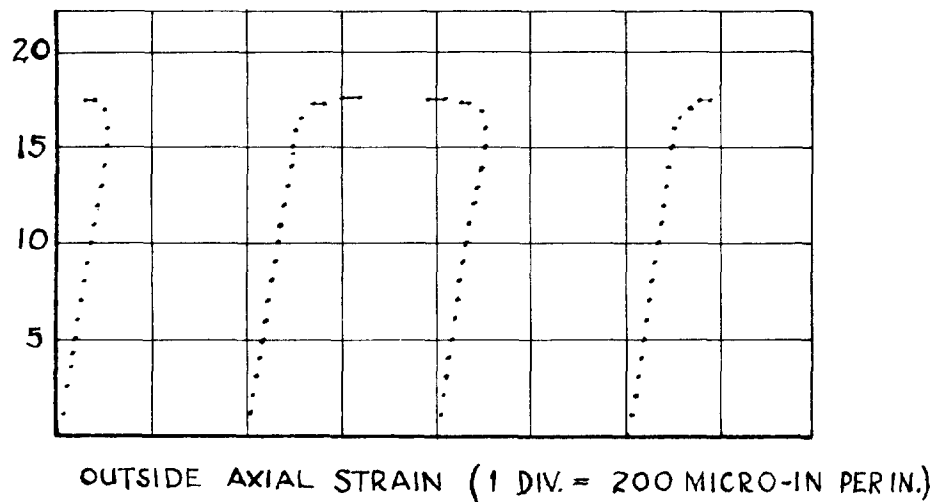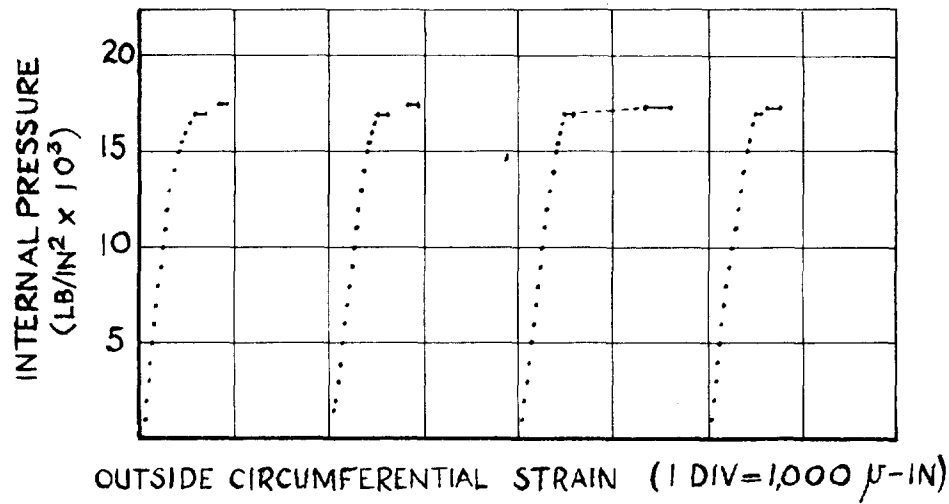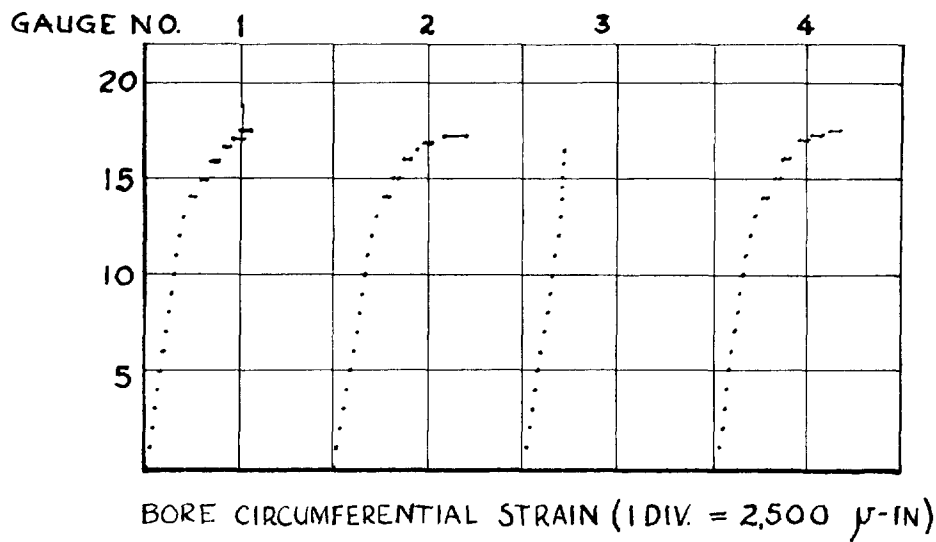
FIG. 5a.   STRAINS MEASURED ON CYLINDER NO. 1

GAUGE NO.　　1　　　　2　　　　3　　　4　　　　5　　　　6

BORE CIRCUMFERENTIAL STRAIN   (1 DIV. = 2500 MICRO-IN. PER IN.)

OUTSIDE CIRCUMFERENTIAL STRAIN   (1 DIV. = 1000 MICRO-IN. PER IN.)

OUTSIDE AXIAL STRAIN   (1 DIV. = 200 MICRO-IN. PER IN.)
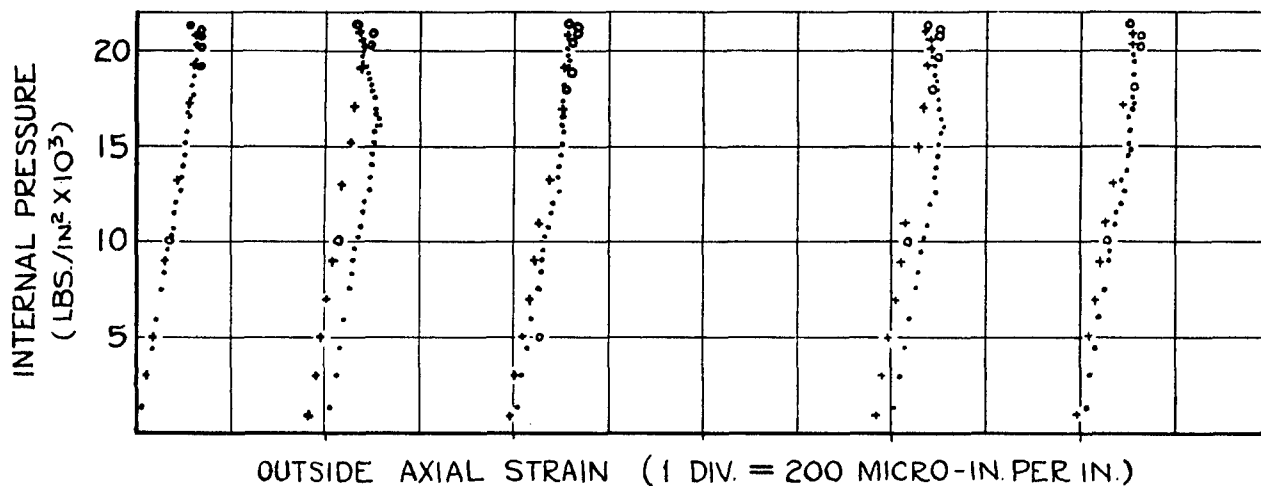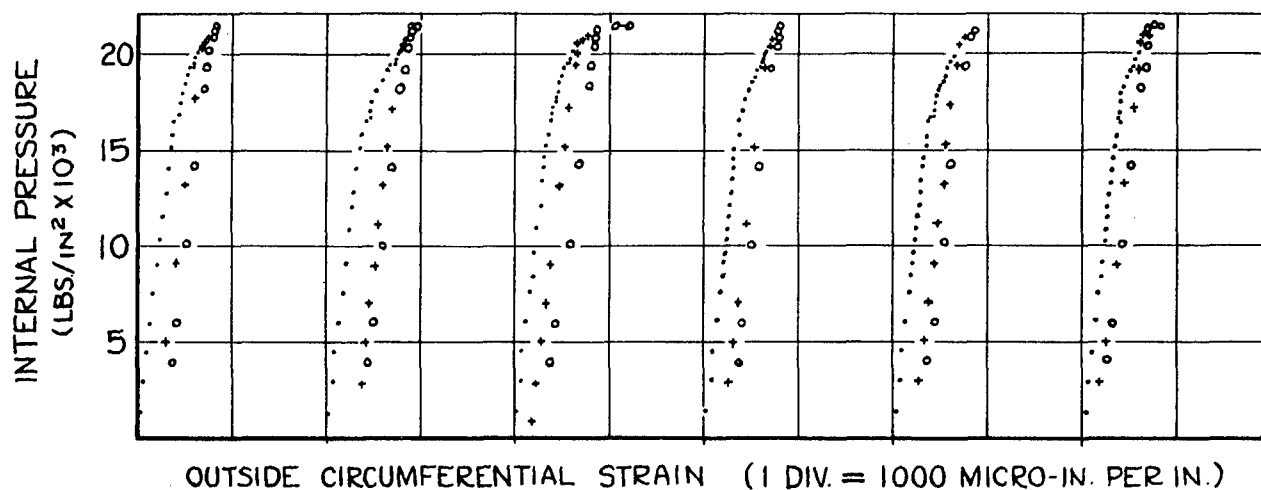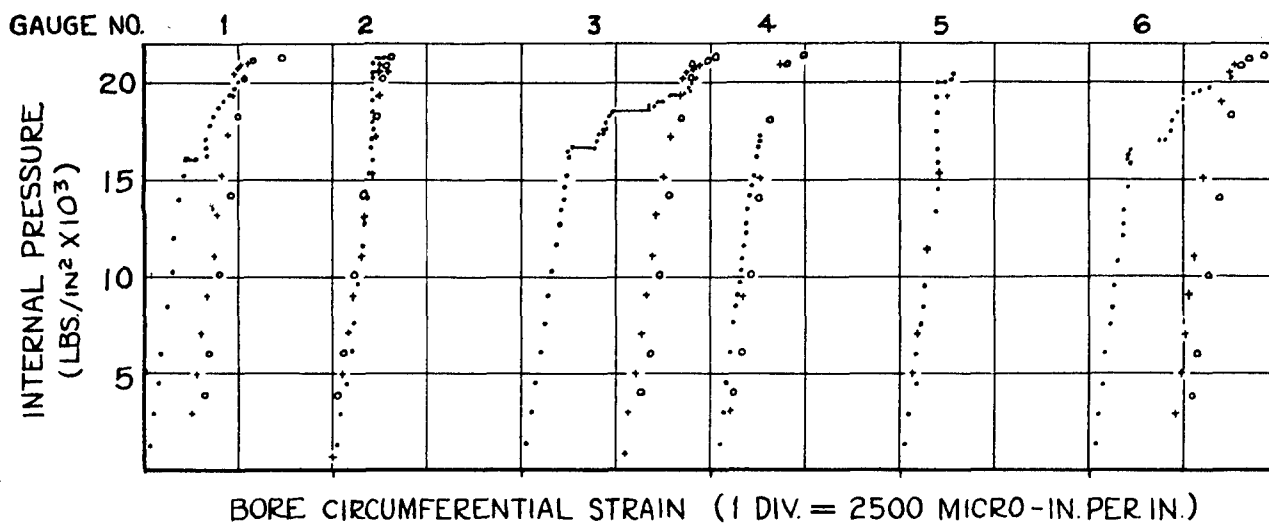
•——1ST LOADING　　　+——2ND LOADING　　　○——3RD LOADING

FIG. 5 b.　　　STRAINS MEASURED ON CYLINDER NO. 2
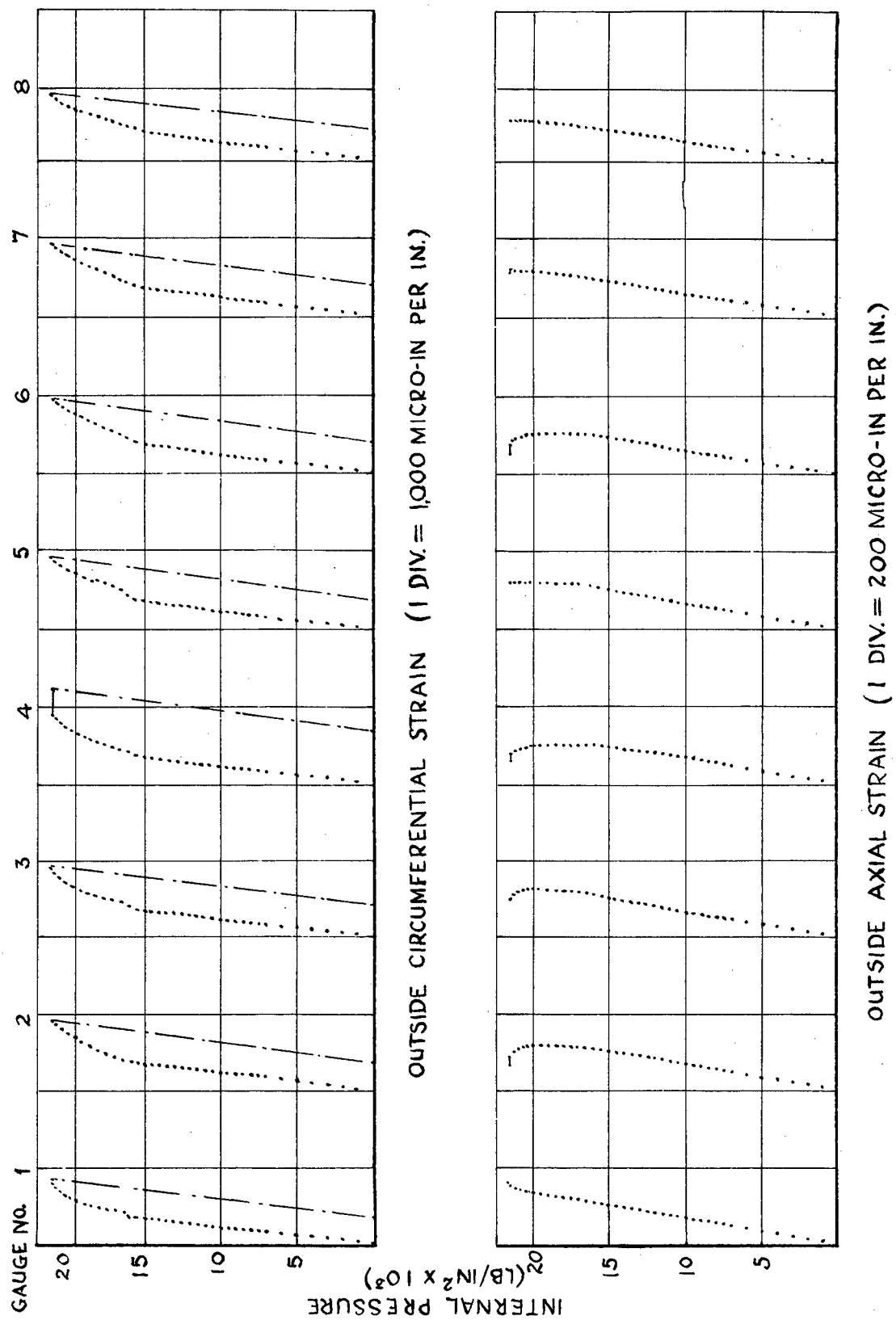
FIG. 5c.   STRAINS MEASURED ON CYLINDER NO. 3
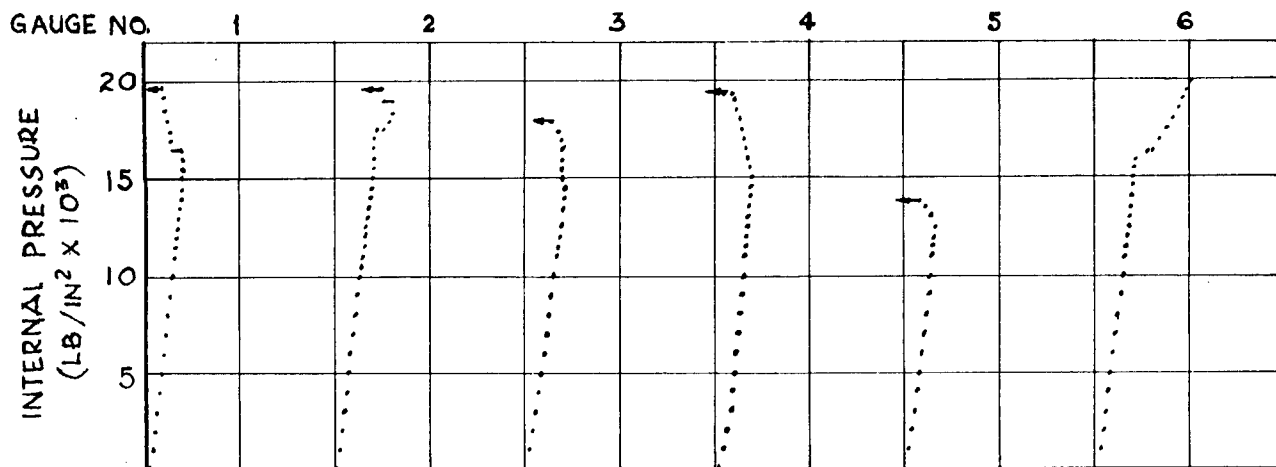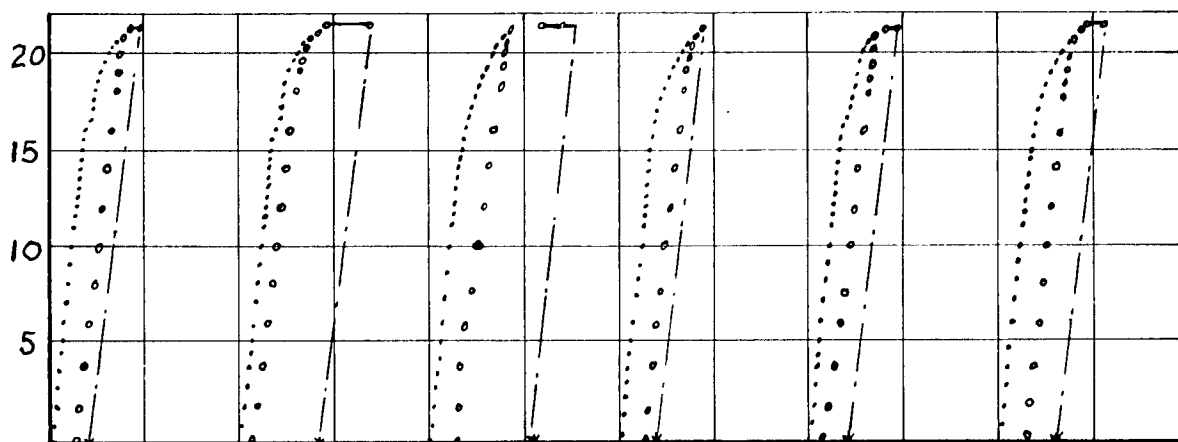
GAUGE NO.   1   2   3   4   5   6

BORE CIRCUMFERENTIAL STRAIN (1 DIV. = 2500 MICRO-IN. PER IN.)

OUTSIDE CIRCUMFERENTIAL STRAIN   (1 DIV. = 1,000 MICRO-IN. PER IN.)
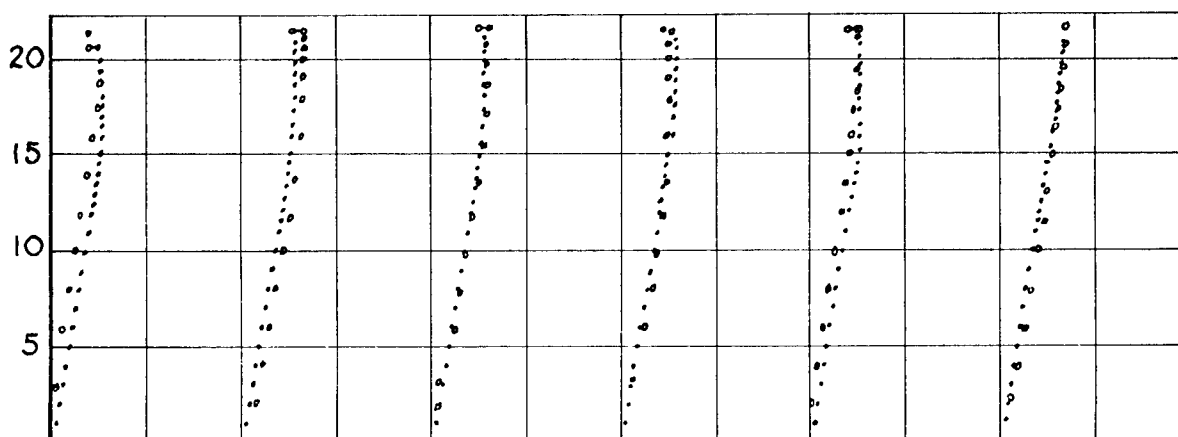
OUTSIDE AXIAL STRAIN (1 DIV. = 200 MICRO-IN. PER IN.)

• — 1ST LOADING   * — FINAL ZERO.   ○ — 2ND LOADING

FIG. 5d.   STRAINS MEASURED ON CYLINDER NO. 4

TOP FACE   BOTTOM FACE

CYLINDER NO. 2 — FULLY PLASTIC

CYLINDER NO. 3 — FULLY PLASTIC

CYLINDER NO. 4 — PARTLY PLASTIC

THEORETICAL DEPTH OF
HOMOGENEOUS YIELD

CYLINDER NO. 4 — FULLY PLASTIC
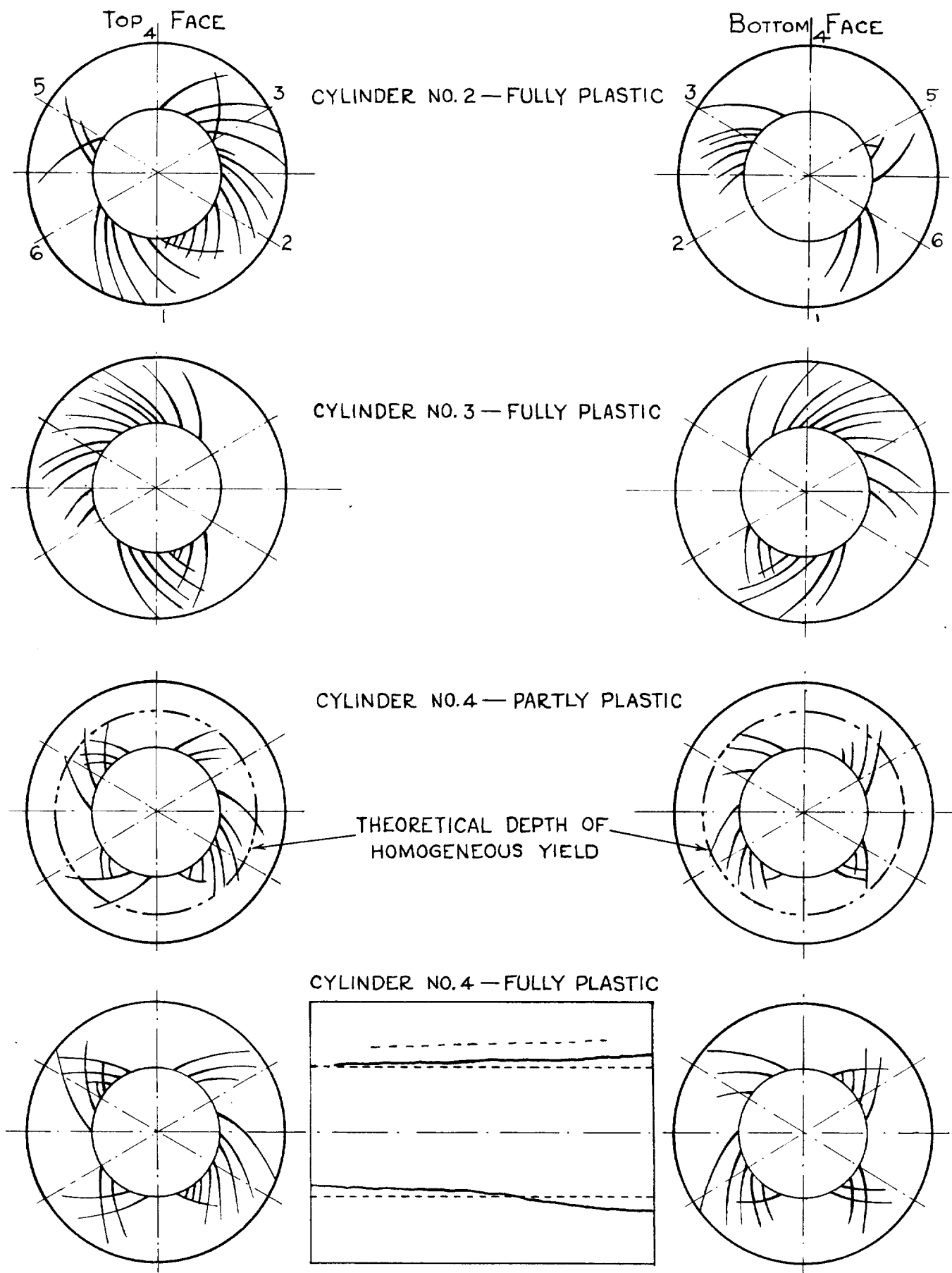
FIG. 6   LOCATION OF WEDGE REGIONS OF OVERSTRAIN

VARIATION OF BORE CIRCUMFERENTIAL STRAIN
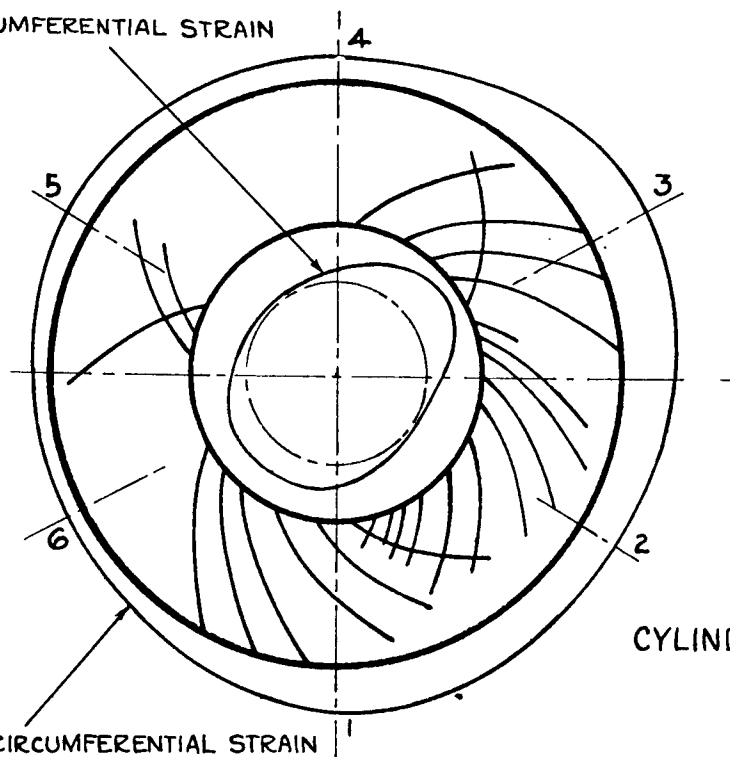
CYLINDER NO. 2

VARIATION OF EXTERNAL CIRCUMFERENTIAL STRAIN

CYLINDER NO. 4

PARTLY PLASTIC

FULLY PLASTIC
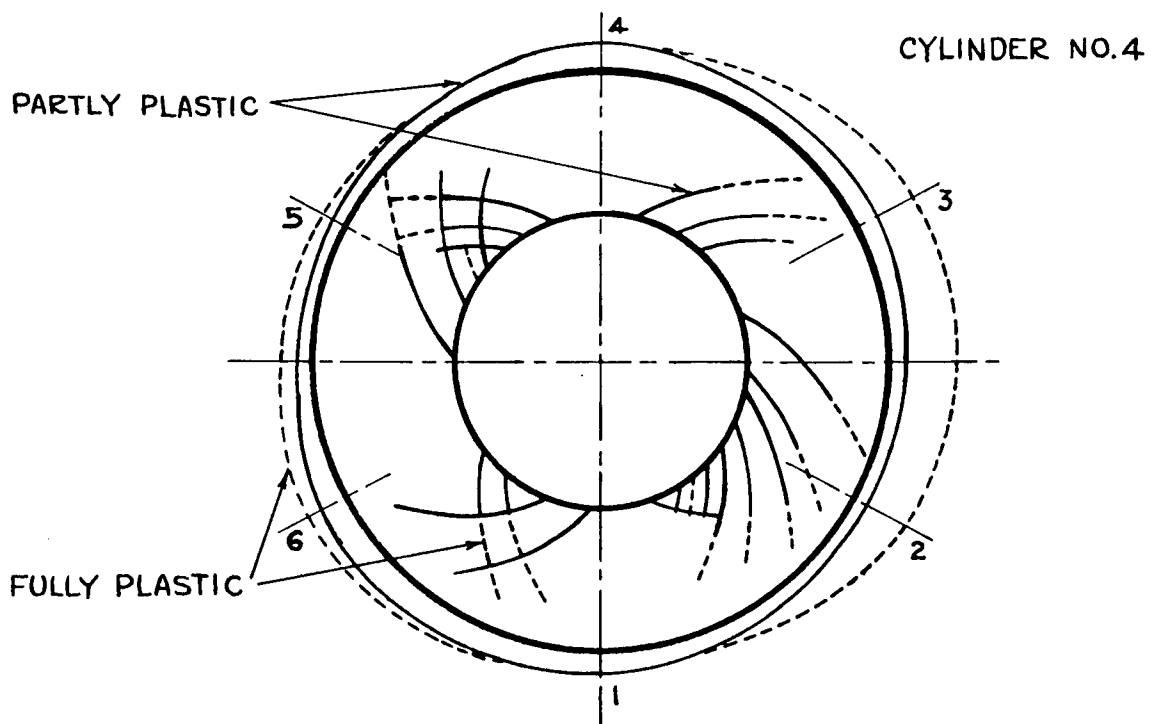
FIG. 7   COMPARISON OF WEDGE REGIONS WITH STRAIN VARIATIONS
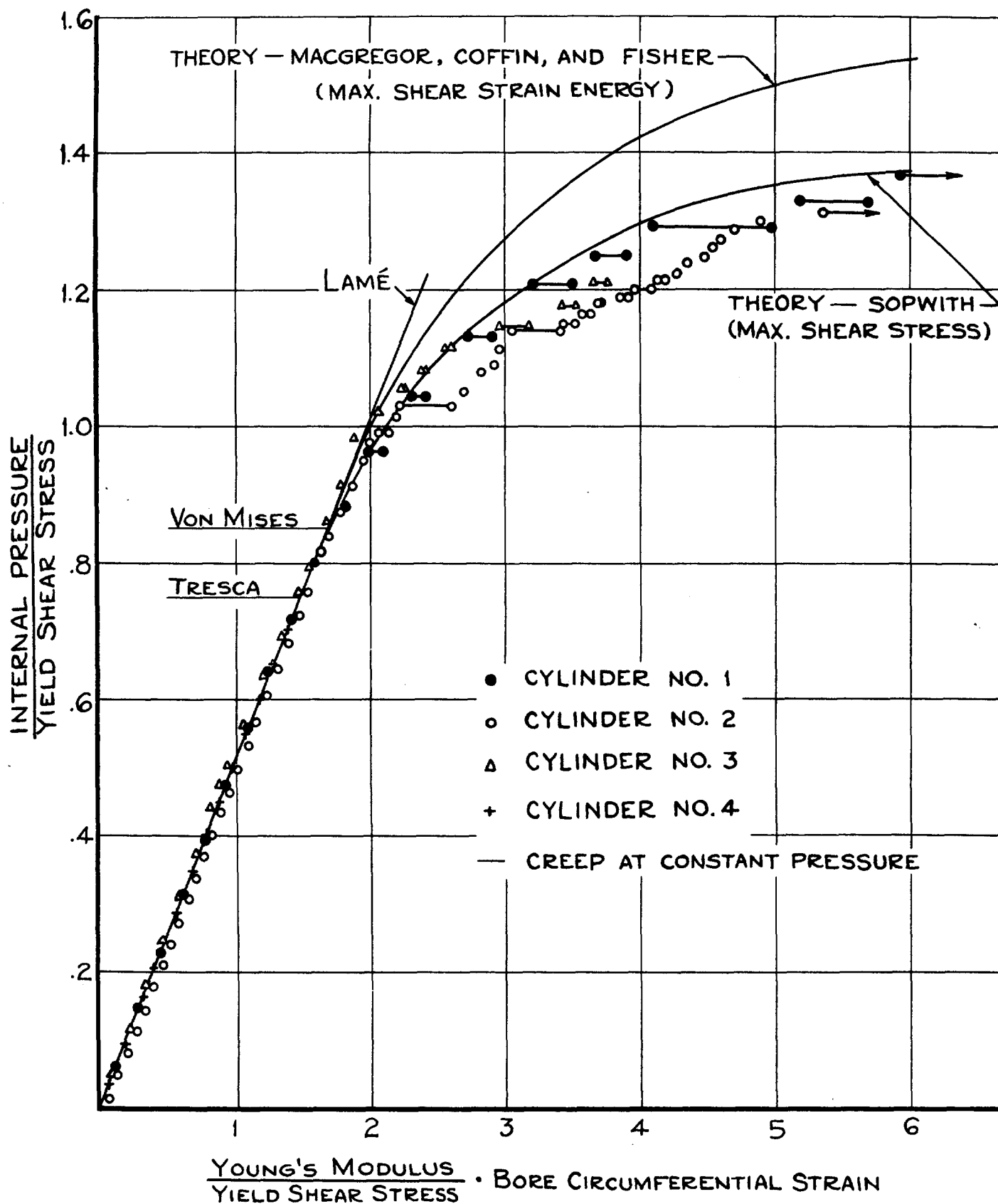
FIG. 8   COMPARISON OF AVERAGE TEST RESULTS OF BORE
CIRCUMFERENTIAL STRAINS WITH PLASTIC THEORIES
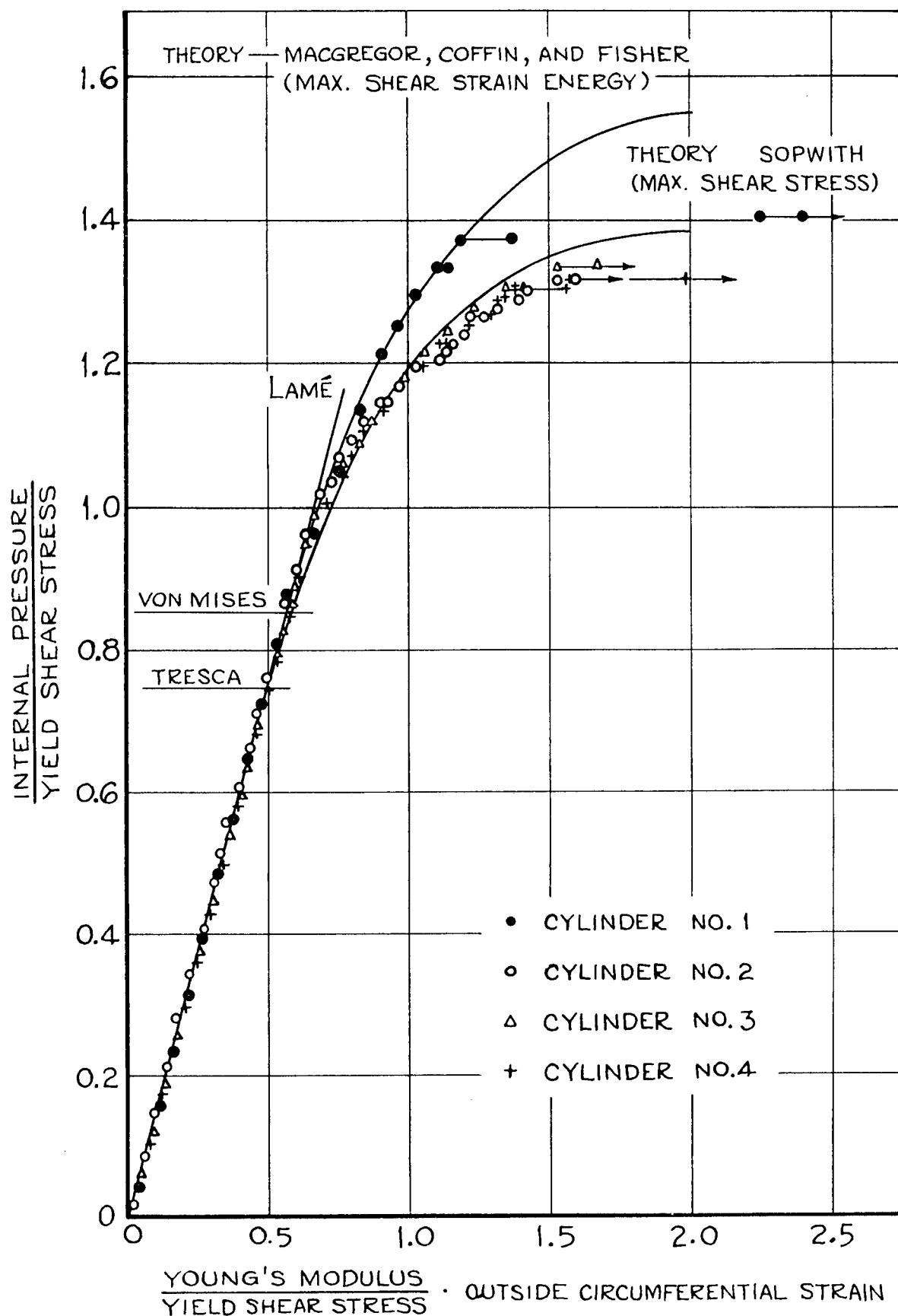
FIG. 9   COMPARISON OF AVERAGE TEST RESULTS OF OUTSIDE
CIRCUMFERENTIAL STRAINS WITH PLASTIC THEORIES
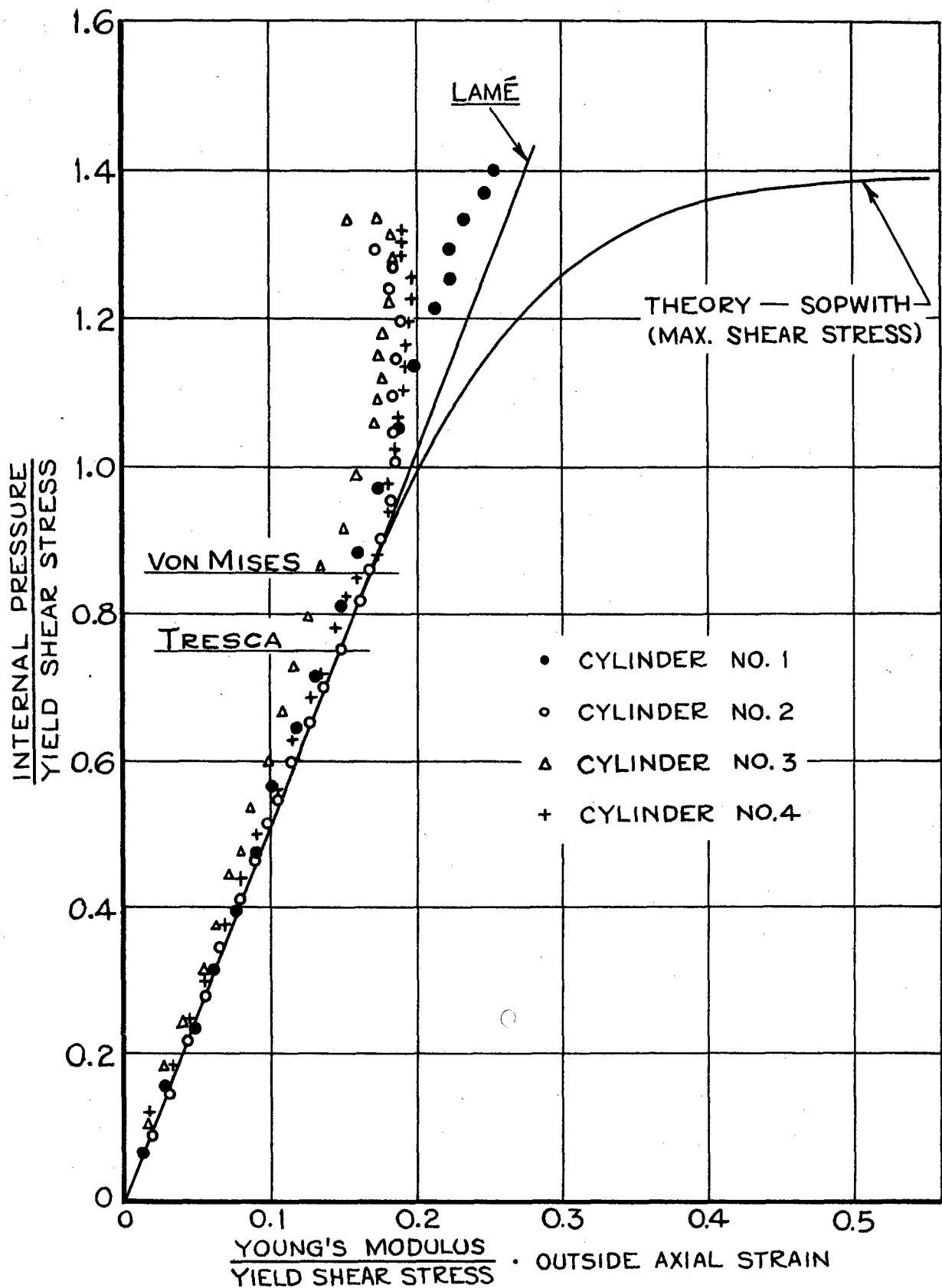
FIG. 10   COMPARISON OF AVERAGE TEST RESULTS OF
OUTSIDE AXIAL STRAINS WITH PLASTIC THEORIES
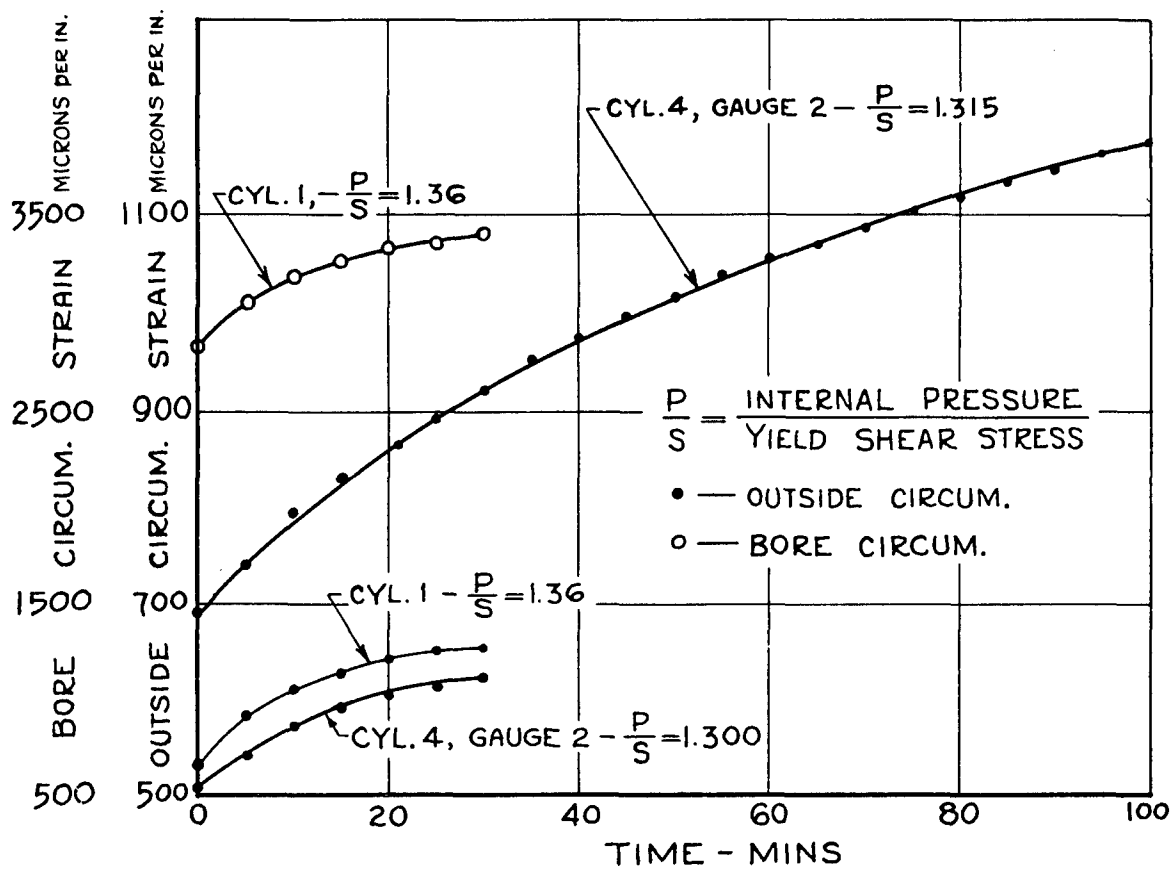
FIG. 11. EFFECT OF CREEP ON CIRCUMFERENTIAL STRAINS
FOR MAINTAINED CONSTANT PRESSURE
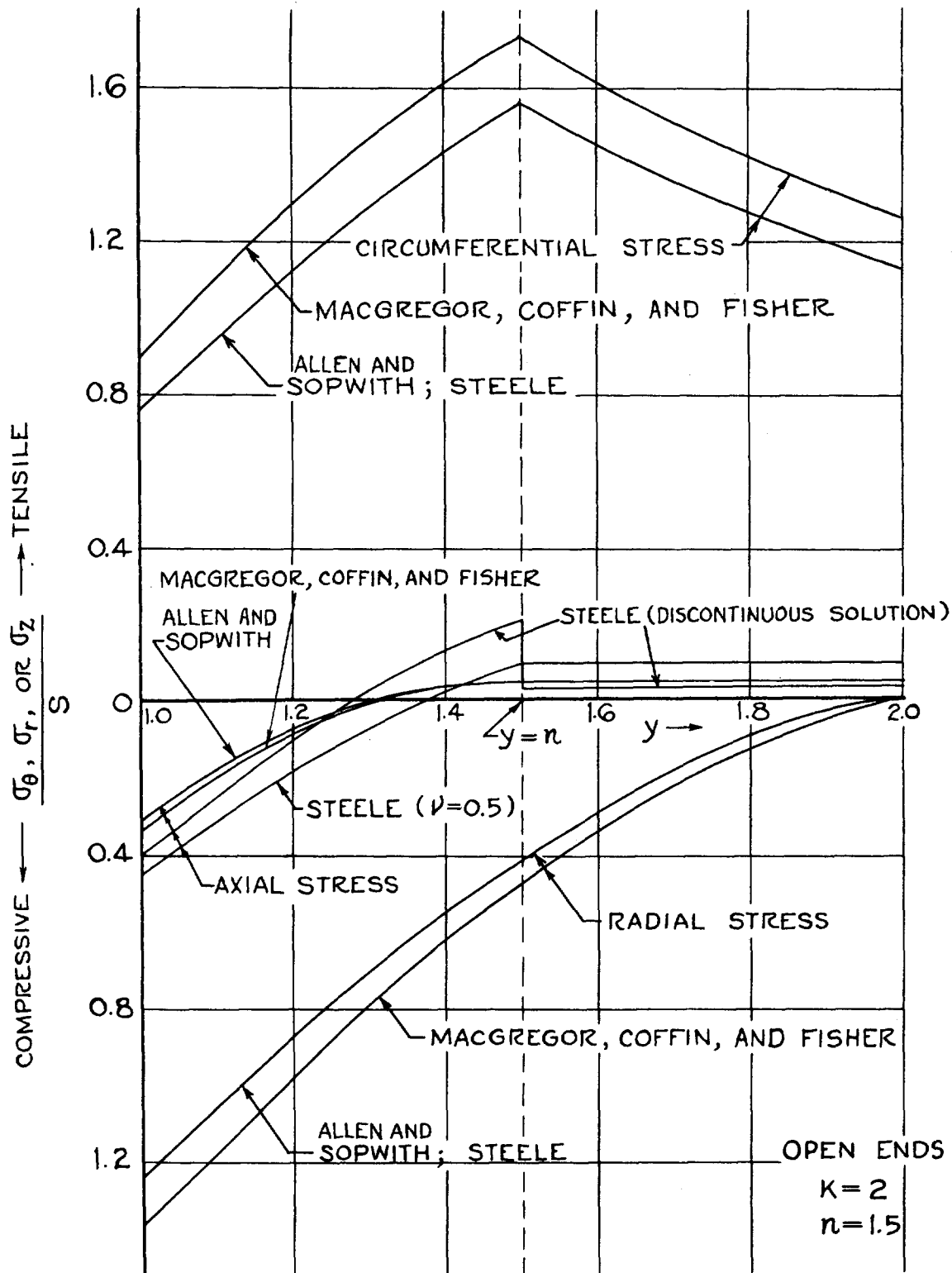
FIG. 12. RADIAL, CIRCUMFERENTIAL, AND AXIAL STRESSES FOR
THEORIES OF (a) ALLEN AND SOPWITH (ν 0.3); (b)
MACGREGOR, COFFIN, AND FISHER (ν 0.3); (c) STEELE
(ν 0.3 AND 0.5)
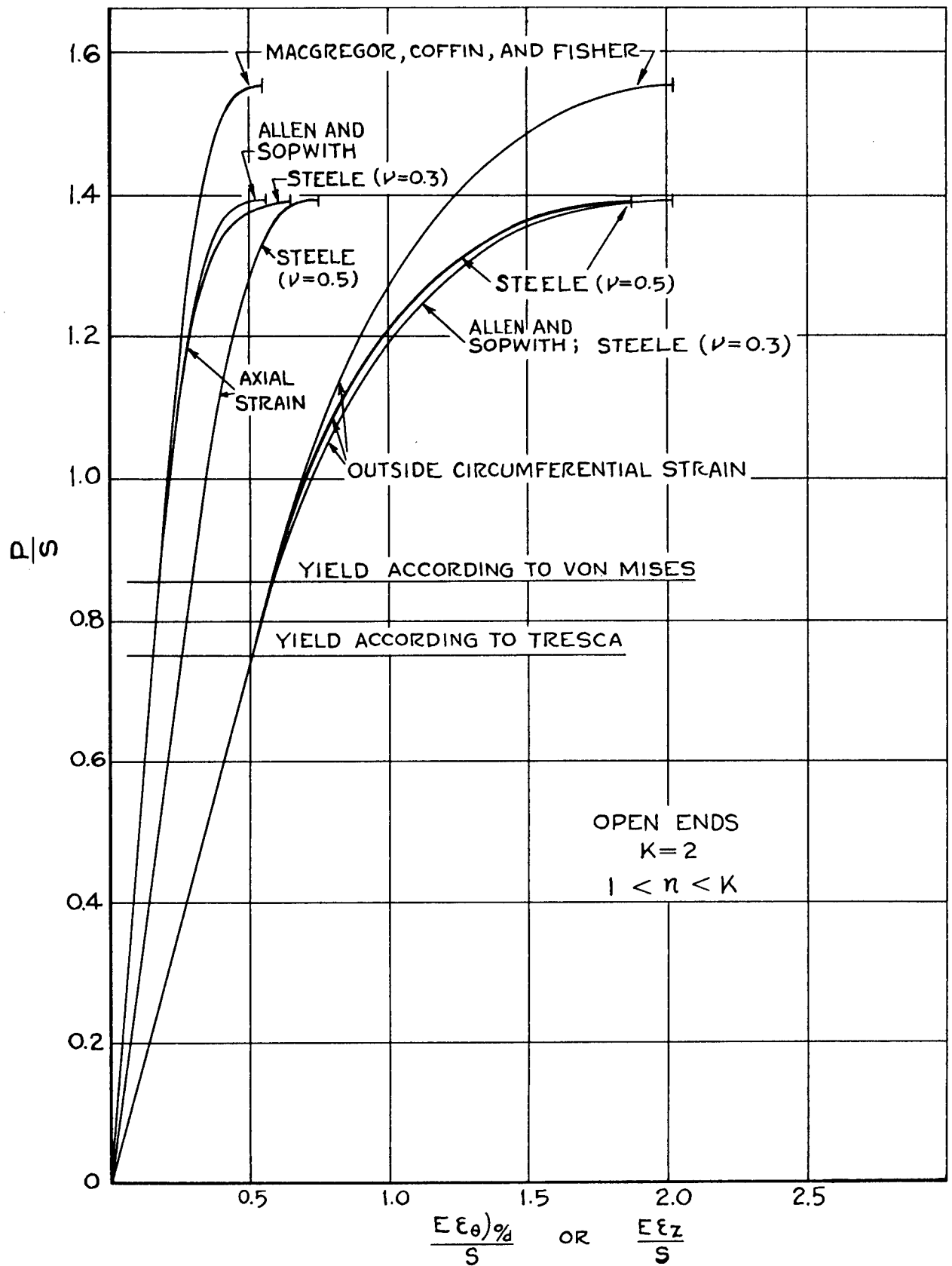
FIG. 13. OUTSIDE CIRCUMFERENTIAL (TENSILE) AND AXIAL (COMPRESSIVE) STRAINS FOR THEORIES OF (a) ALLEN AND SOPWITH (ν 0.3); (b) MACGREGOR, COFFIN, AND FISHER (ν 0.3); (c) STEELE (ν 0.3 AND 0.5)
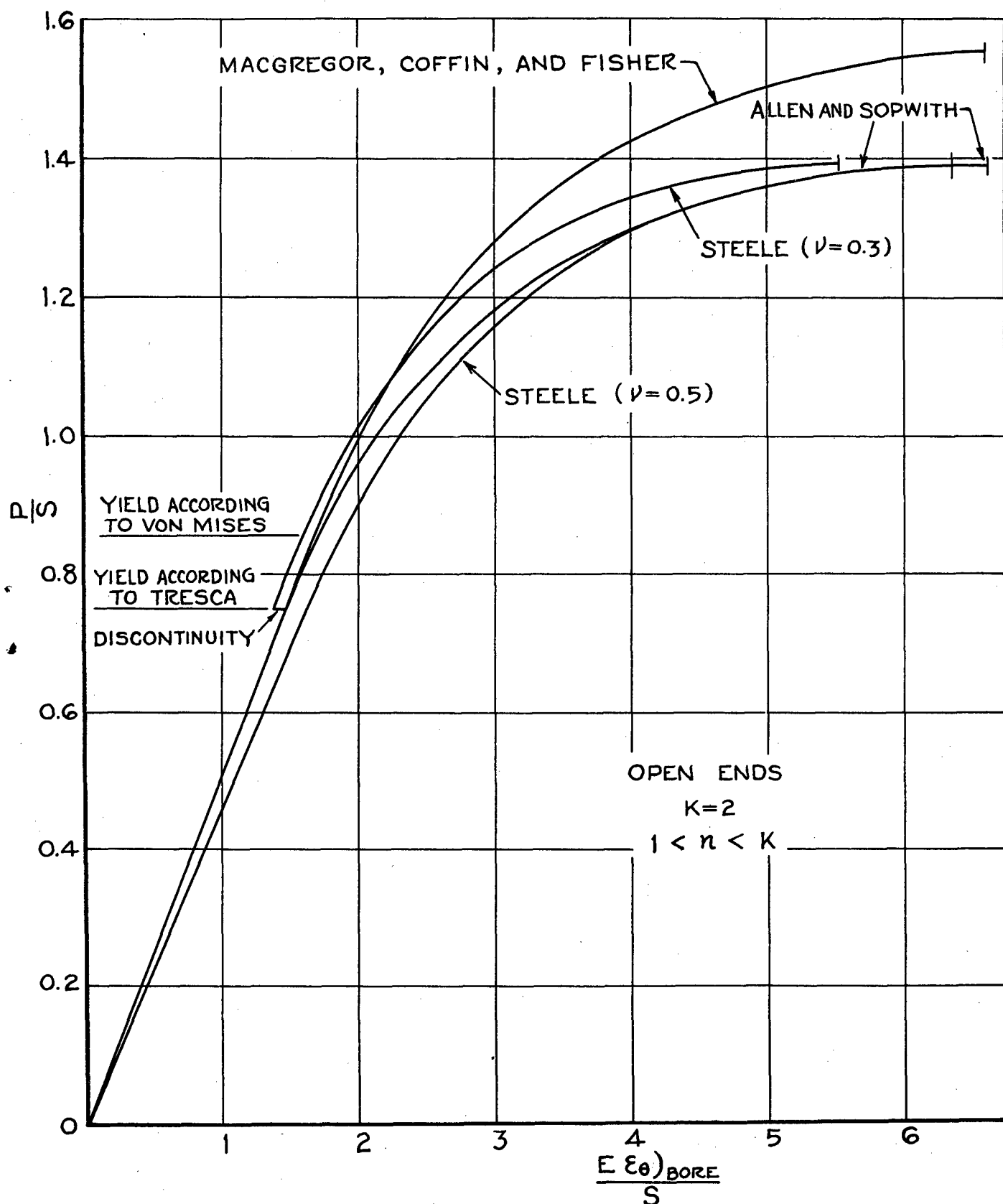
FIG. 14. BORE CIRCUMFERENTIAL STRAINS FOR THEORIES OF
(a) ALLEN AND SOPWITH (ν 0.3); (b) MACGREGOR, COFFIN,
AND FISHER (ν 0.3); (c) STEELE (ν 0.3 AND 0.5)
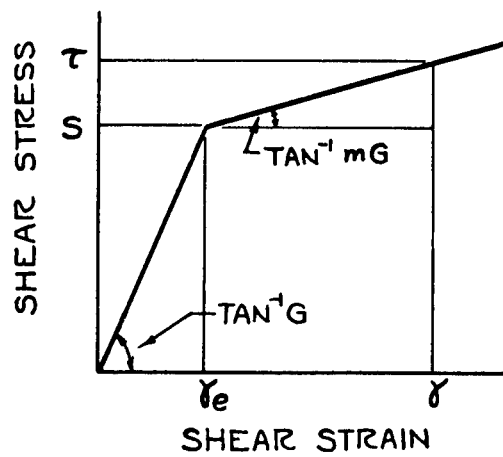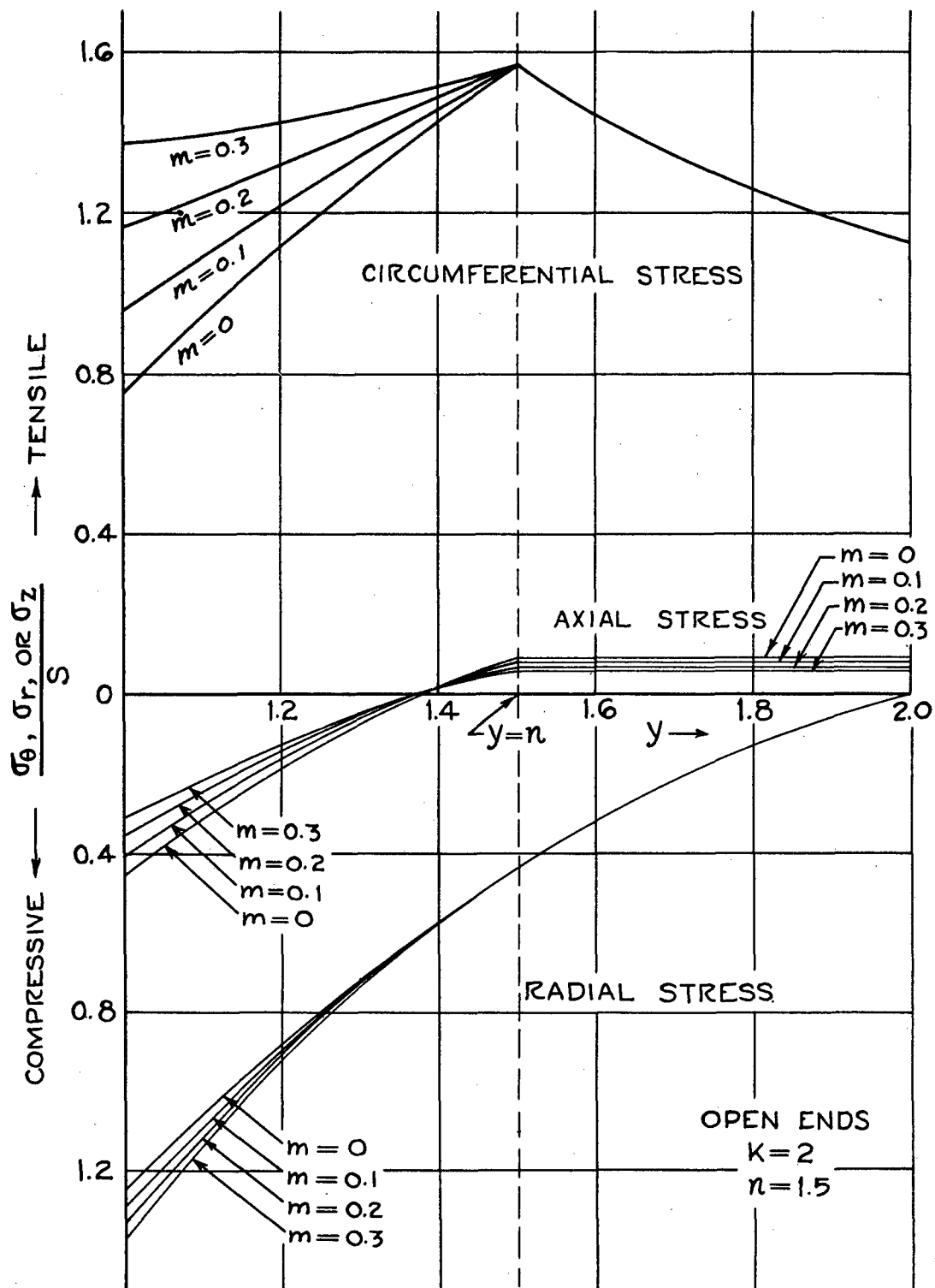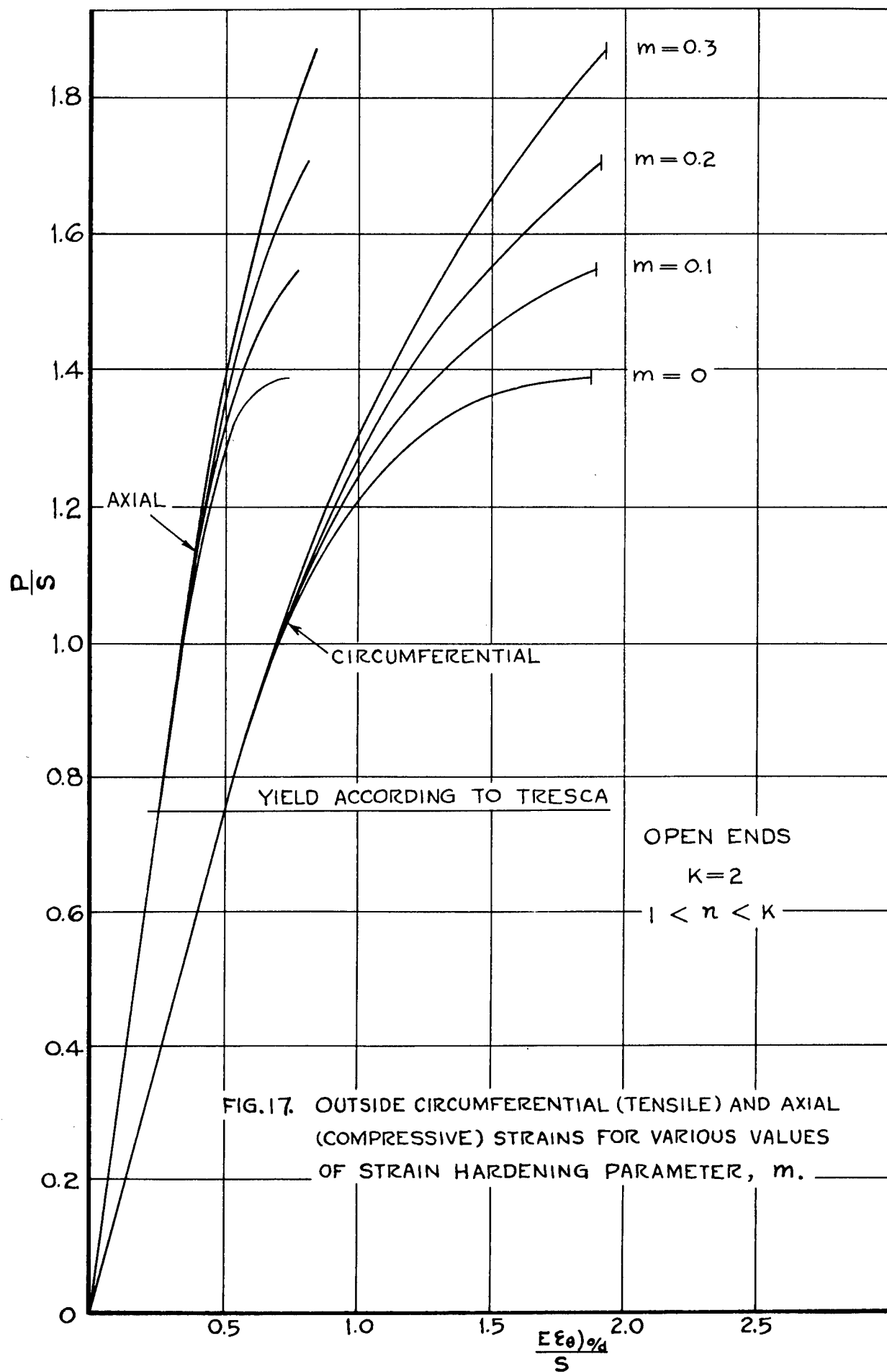
FIG. 15.   LINEAR STRAIN-HARDENING DIAGRAM

FIG.16. RADIAL, CIRCUMFERENTIAL, AND, AXIAL STRESSES FOR
VARIOUS VALUES OF STRAIN-HARDENING PARAMETER, m.

FIG. 17. OUTSIDE CIRCUMFERENTIAL (TENSILE) AND AXIAL (COMPRESSIVE) STRAINS FOR VARIOUS VALUES OF STRAIN HARDENING PARAMETER, m.
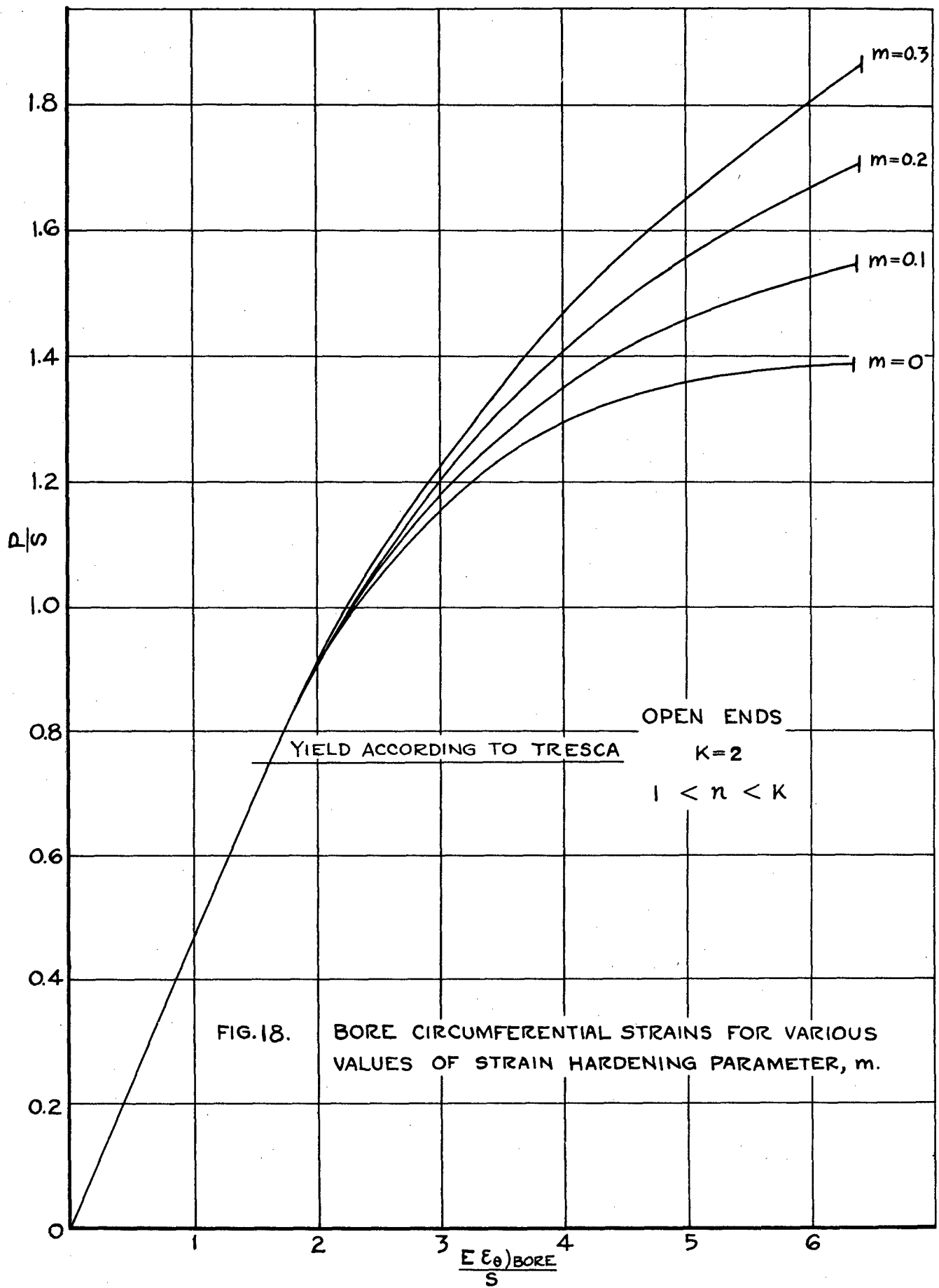
FIG. 18. BORE CIRCUMFERENTIAL STRAINS FOR VARIOUS VALUES OF STRAIN HARDENING PARAMETER, m.
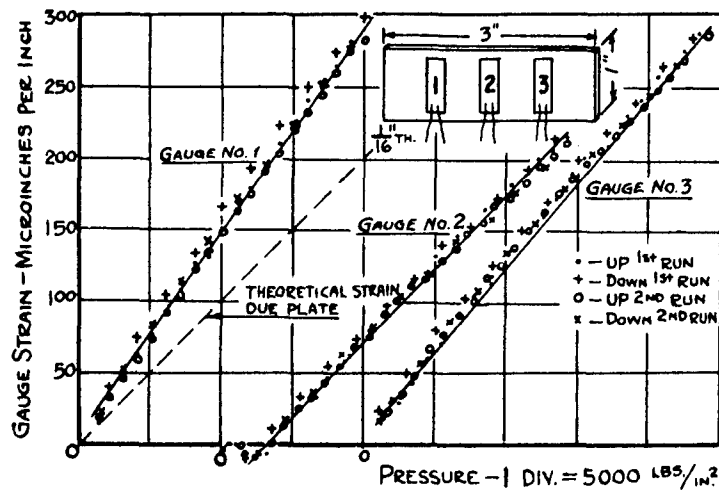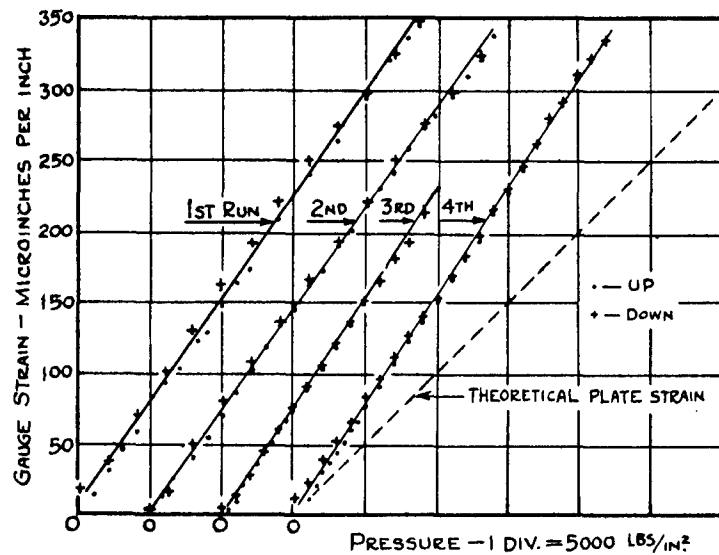
FIG.19. EFFECT OF PRESSURE ON 3 BAKELITE STRAIN GAUGES

FIG.20. REPEAT PRESSURE TESTS ON NO.1 GAUGE